Rates of Growth

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1 Introduction

This paper sets out some important technical results on the meaning and manipulation of growth rates and interest rates in both discrete time and continuous time. Without a basic understanding of the movement of variables over time, and their changes over very short intervals, it would be impossible to think clearly about economic development.

2 Growth Rates: Discrete and Continuous

In economic dynamics, we may use one of two types of analysis: *discrete* or *continuous*.

In the discrete framework, the variables of interest take on a single value per time period (usually, a year). As time advances from Year t to Year $t+1$ to Year $t+2$, etc., the variable's value changes: it jumps discretely at the end of each period to its new value to begin the next period. *Within* periods, the variable does not change at all.

In the continuous framework, the variables are constantly changing at a steady rate through time t.

We illustrate both types of change or growth with the *population of Europe*, which we shall refer to as N. The centralizing concept of growth is very simple: the *annual percentage change* in the variable in question. This is also the principal concept for interest and capital appreciation and we shall discuss these concepts as well. A key word here is "annual": all rates of interest, growth, or change must be defined fundamentally in terms of a precise time dimension. It is almost always a year.

Although we will use real data shortly, assume at first that population is growing at about 10% per year. **Figure** 1 shows the paths of N under both discrete and continuous growth for the first five years, assuming it began at 100 (that is, 100 million people). The actual growth of population in Europe was far smaller.

The smooth curve labeled $N(t)$ is the continuous case. The choppy curve labeled $N_d(t)$ reflects the discrete case. Notice that the two paths both begin at $N(0) = N_0 = 100\,\text{million}$.

The choppy path is the easiest to explain, so we begin there. The formula for N after one year is given by the basic application of percentage change:

$$N_1 = N_0\,(1 + n_d) = 100(1.1) = 110. \qquad (1)$$

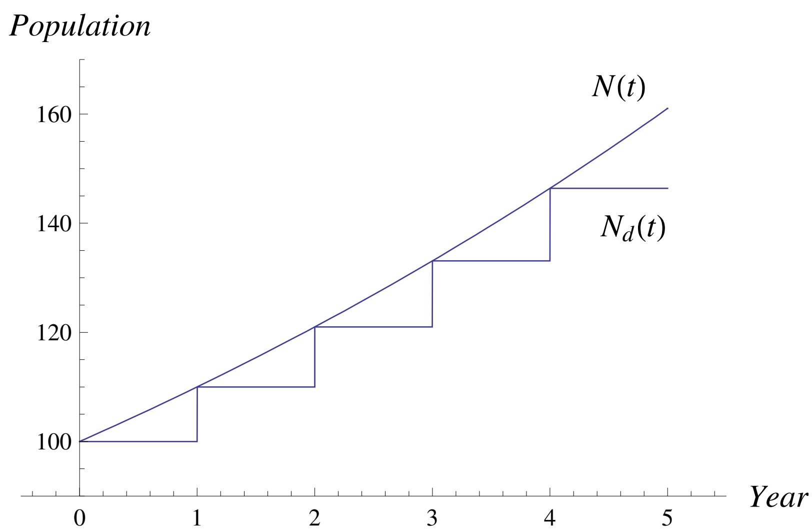

Figure 1: Population Over Time: Discrete and Continuous

In the above expression, n_d is the *geometric growth rate*, which here is .10 or 10%. To find the value for subsequent years, we may apply the above formula repeatedly. For example, for 2 years it is:

$$N_2 = N_1 (1 + n_d) = N_0 (1 + n_d)^2 = 100(1.1)^2 = 121. \tag{2}$$

For t years, where t is any *integer*, we find that population is given by:

$$N_t = N_0 (1 + n_d)^t. \tag{3}$$

We can always calculate the geometric growth rate if we have data for any two adjacent years. This formula is well-known and follows from (3):

$$n_d = \frac{N_t - N_{t-1}}{N_{t-1}}\ . \tag{4}$$

Now consider the smooth growth path also shown in Figure 1. This corresponds to *exponential* or *continuous growth*. In terms of interest, it corresponds to *continuously compounded interest*. The formula for the smooth line is given by:

$$N(t) = N(0)e^{nt} = N_0 \mathrm{Exp}(nt) \ , \tag{5}$$

where n is the *exponential growth rate*. The rates n and n_d are not the same, although they are closely related to one another and n is derived from n_d. The variable e is simply a number. It is a natural constant, like π, and it is given by $e = 2.71828$....... This number never repeats and is not completely known. It is sometimes clearer to write the exponential function as $\mathrm{Exp}(nt)$, but the meaning is exactly the same as e^{nt}

Equation (5) is derived from (1) by cutting the year into very small, equal-sized bits and letting growth (like interest) be compounded *within* the year, not over several years as in (3). For example, if 10% interest were compounded on a monthly basis, and we began with \$100 we would have the following sum at the end of a year:

$$\$W = 100\left(1 + \frac{.10}{12}\right)^{12} = \$110.47. \tag{6}$$

Notice that we get more than just 10%. That is, with compound monthly interest, the corresponding simple yearly rate is 10.47% or, in the decimals that we normally use, .1047. Also, note that the compounding process depends critically on the original *yearly* rate of .10. We divide this figure by the number of periods (e.g. 12 months) before we compound.

If we let the number of periods get infinitely large, we end up with Equation (5). The exponential growth rate n that is based on the basic yearly rate n_d is always smaller than n_d. In fact, the formula is given by:

$$n = \ln(1 + n_d) = \ln(1.1) = .0953102 \ . \tag{7}$$

We discuss the *natural log function* ln(.) below. An exponential growth rate of 9.531% is just enough to keep up with a simple annual growth rate of 10%, which is what is shown in Figure 1.

3 Exponential Growth Rates in Practice

We go through the bother of discussing exponential rates because they are much easier to work with than geometric rates. For this section, recall from high-school math that $x^2x^3 = x^5$ and that $x^3/x^2 = x$. If those results ring a bell, then this section will present no problem.

Consider, for example, the progress of per capita output $y = \frac{Y}{N}$ over time (here Y is GDP), something that we are very concerned about. Assume that Y is growing exponentially at the rate g while N, as before, is growing at the exponential rate n. Then it is straightforward to show that y is growing at the rate $g - n$. Here is how to show it:

$$y(t) = \frac{Y(t)}{N(t)} = \frac{Y_0 e^{gt}}{N_0 e^{nt}} = y_0 e^{(g-n)t} = y_0 \text{Exp}[(g-n)t] \ . \tag{8}$$

We infer by inspection that the rate of change of y is $(g-n)$.

Consider now revenue $R = PQ$: revenue is price times sales. You should be able to work out that the growth rate of revenue is equal to the growth rate of the price *plus* (not times) the growth rate of sales.

Another example concerns a popular production function. Let us say that output depends only on capital and that there are diminishing returns: $Y = K^\alpha$, where $\alpha < 1$. If capital is growing at the rate g_K, how fast is output growing? We derive it as follows.

$$Y(t) = K(t)^\alpha = \left(K_0 e^{g_K t}\right)^\alpha = K_0{}^\alpha \left(e^{g_K t}\right)^\alpha = Y_0 e^{\alpha g_K t}. \tag{9}$$

From this we see that the growth rate of Y is given by $g = \alpha g_K$.

You should be able to show that the growth rate of $Z = \left(\frac{K}{N}\right)^\beta$ is given by: $g_Z = \beta\,(g_K - n)$.

One final problem: find the growth rate of Y if the production function is $Y = K^\alpha L^\gamma$ where L is workers, which differs from population. That is, find g in terms of the growth rates of capital g_K and workers g_L (which may equal n even though $N < L$).

These results are even more important than they might seem because they generalize to *instantaneous growth rates*, no matter what the underlying process that is generating the change. We discuss this in more detail below.

Next, however, we discuss the natural log function.

4 Natural Logs

Everyone knows what a *square root* is. But defining it out loud can be slightly difficult. The reason for that is, I think, that all a square root does is *undo* a square! Without the square of a number, the square root is meaningless. To see this more formally, define the following function that squares a number:

$$S(x) = x * x = x^2 \ , \tag{10}$$

where we may call $S(x)$ the "square function". That is, $S(5) = 25$. Simple enough.

Now define the "square root function" $R(x)$. This function just *reverses* or *undoes* the square function. That is, $R(25) = 5$. It is hard to write down a general function the way we wrote (10) above. Perhaps the most informative way to write it is as follows:

$$S(R(x)) = x \quad \text{and} \quad R(S(x)) = x. \tag{11}$$

That may look odd but it just says what we all know: $\left(\sqrt{x}\right)^2 = x$ and $\sqrt{x^2} = x$.

The reason to bring this up is that the *natural log function* $\ln(x)$ is much the same as the square root. It simply *undoes* the exponential function. That is:

$$\ln\left(e^{x}\right) = x \text{ and } e^{\ln(x)} = x \ . \tag{12}$$

The log function has no other meaning and does nothing else. Nonetheless, it is extremely valuable. One very important property of the log function is that

$$\ln(AB) = \ln A + \ln B. \tag{13}$$

Here is how to show (13). Use (12) to express $AB = e^{\ln A} e^{\ln B}$. But the latter can be written as $e^{\ln A + \ln B}$. Now take the log of the first and last expressions in that sequence. They must be equal, which proves (13). It also follows that $\ln(A/B) = \ln A - \ln B$, another very useful result.

We illustrate the importance of the natural log with the population of Europe in the next section.

5 The Population of Europe

Our best estimates show that Europe's population was 81 million in 1500 AD and rose to 728 million in 2000. We shall simply *assume* or *impose* exponential growth with a constant growth rate from the beginning to the end of the 500 year path. This means that we assume the following is satisfied:

$$N_{2000} = N_{1500} e^{nt}. \tag{14}$$

Notice that we know everything about (14) except the value of n. Thus, $t = 500$, $N_{1500} = 81$, and $N_{2000} = 728$. To find n, we substitute in the values, then take the natural log of both sides of (14) to get:

$$\begin{aligned} \ln 728 &= \ln 81 + n500 \implies \\ & n = \frac{1}{500}(\ln 728 - \ln 81) = .0043917 \text{ or } .439\% \ . \end{aligned} \tag{15}$$

In other words, the population of Europe grew, *on average*, at less than $\frac{1}{2}\%$
.

Figure 2 shows our hypothesized, exponential path for population. Of course it was not that smooth: war, famine, and disease still wreaked havoc on the European population even after 1500. Only the first and last points are “known”: the others are generated to make the path smooth.

Figure 3 shows the path of the *natural log of the population* over this time period (again, our hypothesized path). It is given by

$$\ln N = \ln 81 + .00439t$$

The interesting thing here is that the *natural log* of a variable that is growing exponentially in nature is a *straight line*. It is amazing how many economic

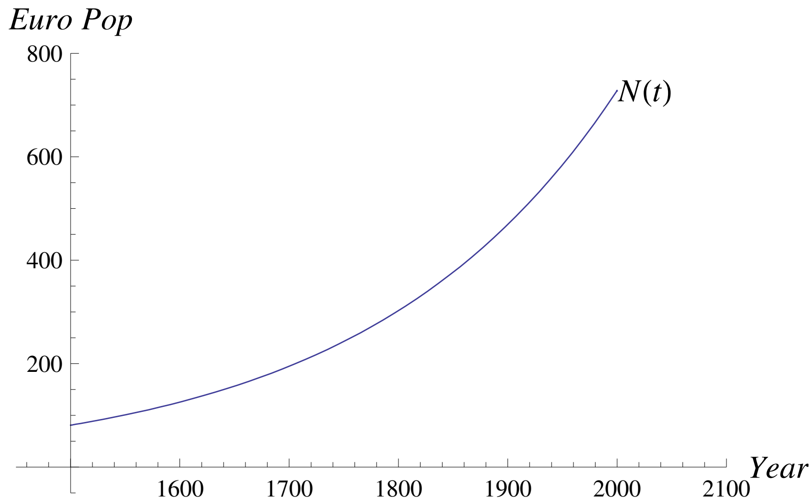

Figure 2: Population of Europe

time series when transformed into the natural logarithm (like the natural log of US GDP per capita) are straight lines over many years.

6 Negative Exponential Growth

Nothing prevents the growth rate from having a negative value. For example, if population were growing faster than GDP, the growth rate of y would be negative. You can see that this is a possibility from Equation (8). Assume that $g = .02$ and $n = .04$, which is large, but not out of the question. Figure 4 shows what the path of y through time would look like. It is amazing how quickly $y(t)$ falls to near-zero at only a 2% annual decline. Luckily, such long-run declines are extremely rare, even with war and very unstable government.

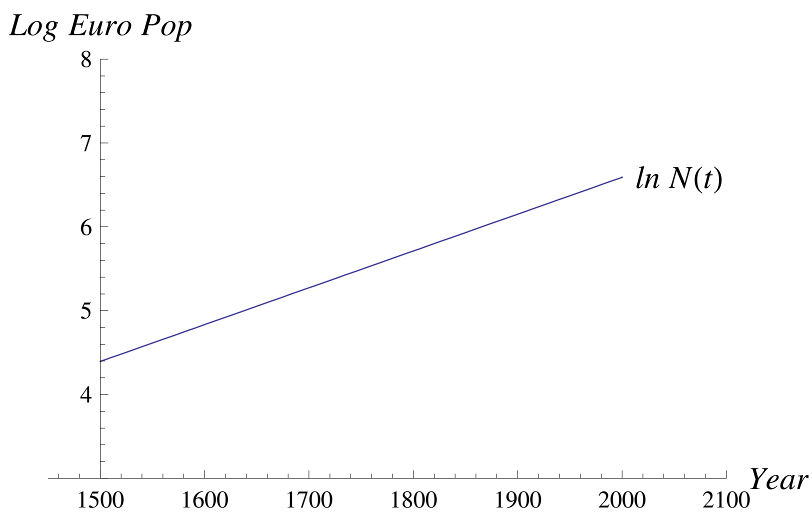

Figure 3: Log of European Population

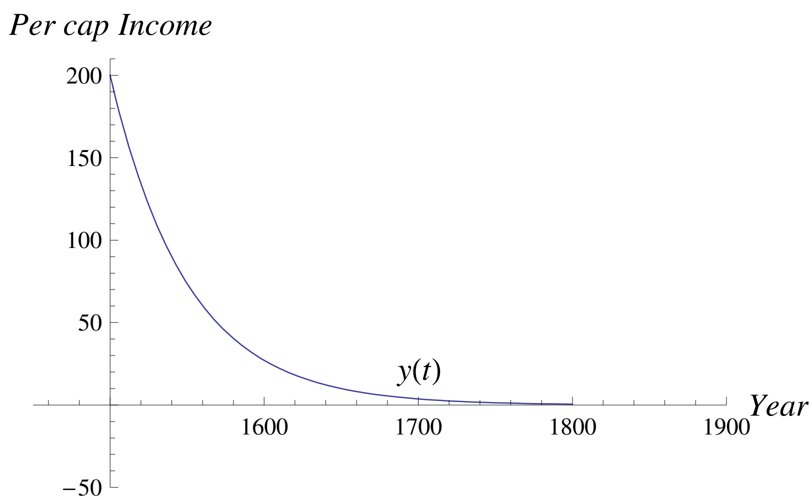

Figure 4: Negative Exponential Growth

7 Instantaneous Growth Rates

So far, we have treated rates of change over very long intervals, seeking to find the average annual growth rate – either discrete (also called geometric) or exponential (also called continuous). Now we focus on a smaller time period so that the results pertain to *any* variable moving through time.

First, let's consider a *yearly percentage change* in the population. From (4) and (15) – with $t = 1$ – we may say that the two rates n_d and n are *approximately* equal. Let us write them as follows:

$$n_d = \frac{N_{1996} - N_{1995}}{N_{1995}} = \frac{\Delta N}{N} \approx n = \ln N_{1996} - \ln N_{1995} = \Delta \ln N \ . \qquad (16)$$

The symbol "Δ" means "change in" over a specific time period. The squiggly equals sign " $\approx$ " means "approximately equal to".

What if we were interested in the *yearly* percentage change over a smaller interval, like 6 months? Again, everything is based on a yearly rate. This is very important. To find the answer, we would "pro-rate" the percentage change by dividing through by the *fraction of the year* over which the change takes place. We call this *fraction* "Δt" . Thus, we re-write (16) as:

$$n_d = \frac{\Delta N}{\Delta t}\frac{1}{N} \approx n = \frac{\Delta \ln N}{\Delta t} \ . \qquad (17)$$

The above is very general, since for a year $\Delta t = 1$. For 1 month, $\Delta t = \frac{1}{12}$;

for a day, $\Delta t = \frac{1}{365}$. The change in population is accordingly measured only over the month or day as the case may be. By dividing this way, we keep expressing the change on an annual basis.

Although we cannot show it here, something very interesting happens as the time interval gets short: the approximation gets better. For a very short – infinitesimally short – interval, the two are exactly the same! We use the symbol "d" in place of "Δ" to refer to this extremely short time interval. Thus, we may now write:

$$n = \frac{dN}{dt}\frac{1}{N} = \frac{d\ln N}{dt} \ . \tag{18}$$

Perhaps an example will help clarify the basic idea. Let's say that at the very beginning of the day of July 16, 2005 the population of the Czech Republic was $N = 10.24$ million. On that day, the net increase in the population was 420 people (not millions!). First, note that the first term in (18) can be written as:

$$n = \left(\frac{420}{\frac{1}{365}}\right)\left(\frac{1}{10.24 * 10^6}\right) = 0.0149707$$

How do we interepret this? It is the percentage by which the Czech population *would have grown over the year*, if the rate of 420/day had continued. Again, note that the *year* is the key. This is about 1.50% growth rate for the Czech population.

The first ratio of (18), $\frac{dN}{dt}$, is called the *time derivative* or *rate of change* in population and it is very important in growth theory. Notice that it is closely related to the *growth rate* or *percentage rate of change*, n. They differ only by the factor N. Often, we use the "dot" notation for the time derivative. That is we would write (18) as:

$$n = \frac{\dot{N}}{N} \implies \dot{N} = nN \ . \tag{19}$$

Again, the meaning of $\dot{N}$ is the absolute change in N over a small interval, *expressed on a yearly basis*.

What about the second expression in (18), the log form? The change in the log of N can be written as $\ln[(N + \Delta N)/N]$. This uses (12) above. Here is the math for calculating the growth rate using logs:

$$n_2 = \ln \left(\frac{10.24 * 10^6 + 420}{10.24 * 10^6} \right) * 365 = 0.0149704$$

Notice that the two rates are extremely close. This is because our time interval – a day – is very short. If we did a calculation for a *minute*, the two numbers would be even closer.

Although I have used population here, the ideas are applicable to any economic time series, such as GDP, capital, or the price level. In the case of capital K for example, we have:

$$g_K = \frac{\dot{K}}{K} \implies \dot{K} = g_K K \ . \tag{20}$$

It is easy to go back and forth between growth rates and time derivatives. In the next chapter we will employ these techniques to discuss the fundamental model of economic growth.

8 Interest Rates

Interest rates, like growth rates, can be expressed either as simple discrete (geometric) rates, rates that are compounded at various terms – see (6) – *or* as continuously compounded (exponential) rates.

For example, if you begin with $\$A_0$ in the bank and keep it there for t years at simple interest of R_d, at the end of the t years you will have:

$$\$A_t = \$A_0 \left(1 + R_d\right)^t \ . \tag{21}$$

The above is strictly only valid if t is an integer (i.e. 1, 2, 3, ...).

If the interest rate were instead compounded continuously, then the money would grow to:

$$\$A(t) = \$A_0 e^{Rt} \tag{22}$$

after t years. Here, there is no requirement that t be an integer.

To take a simple example, assume that you begin with \$100.00 and can earn $R_d = .05$ simple interest in Bank 1 and $R = .05$ continuously compounded interest in Bank 2. In Bank 1, after 10 years you have \$100(1 +

$.05)^{10} = \$162.889$, In Bank 2 you have: $\$100 * \text{Exp}[.05 * 10] = 164.872$.

Clearly, continuous interest *of the same magnitude* is better.

9 Present Value

Suppose you will receive $\$B$ in year T. What is that worth *today?* You could say it is worth the amount $\$A$, such that, with interest, $\$A$ will grow to $\$B$ in T years. As before, we may analyze the question in either discrete or continuous time.

In discrete time, A must satisfy:

$$A(1+R)^T = B \implies A = \frac{B}{(1+R)^T} \ . \tag{23}$$

In continuous time, A must satisfy:

$$Ae^{RT} = B \implies A = Be^{-RT} \ . \tag{24}$$

Here is a simple problem. Find the present value of \$100 to be received in 10 years, if $R = .03$. Find both the continuous-time value and the discrete-time value.

According to (23): $A = \frac{100}{(1.03)^{10}} = \$74.41$.

According to (24), we find: $A = 100e^{-.03*10} = \$74.08$

At very low interest rates, the two are very similar.

10 Annuities

Let's say you had \$50,000 today. What *constant amount* could you consume *each year forever* with that amount? The answer is what we call the "annuity value" of the principal.

To find the answer, we note that in continuous time, the initial amount must equal the integral of the present value of each year's constant consumption, C. That is:

$$A = \int_0^\infty Ce^{-Rt}dt \implies C = RA \tag{25}$$

The formula is very simple and intuitive: you can consume the interest on the asset forever.

In discrete time, the condition is still the sum of present values, which leads to:

$$A = \sum_{t=0}^{\infty} C\left(\frac{1}{1+R}\right)^t \implies C = \left(\frac{R}{1+R}\right)A \tag{26}$$

If Jane has an inheritance of euro 250,000, how much could she consume forever given that the interest rate is 10%? Find the answer in both continuous and discrete terms.

In continuous terms:

$$C = .10 * 250,000 = 25,000 \tag{27}$$

In discrete terms:

$$C = \left(\frac{.10}{1.10}\right) 250,000 = 22,727.30 \ . \tag{28}$$

11 Conclusion

These are important techniques for use in all advanced economics and finance courses. After working with them for a while, it becomes much easier to use them and understand why they are so valuable.