

# Early Development: Mathematics and Reader's Guide

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## 1 Introduction

In the paper Early Development, there are three activities. The effort devoted to each must sum to 1:

$$1 = e_P + e_M + e_L \tag{1}$$

Here,  $e_P$  is work in the primitive sector,  $e_M$  is work in the market (urban) sector, and  $e_L$  is learning effort.

## 2 Primitive Technology

There are  $f$  villages, each of which has  $n$  people. So the economy's population is  $N = nf$ . Village population grows at the rate  $\eta$ ; the number of villages  $f$  does not change.

Total village output is:

$$Y_P = B \ln(1 + e_P n) \tag{2}$$

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so that per capita output is:

$$y_p = \frac{Y_P}{n} = \frac{B}{n} \ln(1 + e_P n) \quad (3)$$

And the marginal compensation for extra work is:

$$w_P = \frac{\partial y_P}{\partial e_P} = \frac{B}{(1 + e_P n)} \quad (4)$$

This is one of the two lines in the main figure of the paper. It is like a wage, even though there is no labor market in the primitive sector.

### 3 Technology and Market Structure

#### 3.1 Final-Good Technology

Two goods are produced in a closed economy with a technology that uses a variety of specialized input goods – produced in the first stage – combined with labor augmented by human capital in the second stage. This production function is:

$$Y = (e_Y h_a N)^{1-\alpha} \int_0^M (x(i))^\alpha di \quad (5)$$

The variable  $Y$  is total output of this sector and  $e_Y h_a N$  is the effective labor engaged in the second stage to produce it. The variable  $e_Y$  represents the skilled effort used to process specialized goods into the final good.  $N$  is the population and  $h_a$  is *average* individual human capital.<sup>1</sup> The variable  $x(i)$  is the quantity of intermediate good  $i$  used. The limit  $M$  refers to the range of intermediate goods that are used to produce this good. We consider the first-stage production below.

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<sup>1</sup>Assume that human capital is distributed according to  $f(h)$ . That is,  $f(h)$  is the number of people with human capital level  $h$ . Then  $N = \int_{h_{min}}^{h_{max}} f(h) dh$ . Aggregate human capital is given by  $H = \int_{h_{min}}^{h_{max}} h f(h) dh$  and the average is given by  $h_a = \frac{H}{N}$ . If everyone were alike and had human capital  $\check{h}$ , which will be our assumption, then  $H = \check{h}N$ . The important point is that individuals take the aggregate amount of human capital  $H$  and the average  $h_a$  as given when they formulate their optimal plan below. We could make the same argument about  $e_Y$  but there is no need, since the external effect here does not enter into the accumulation decision.

Scale raises output per capita with this technology, as in Romer (1987) and Goodfriend and McDermott (1995, 1998). These scale economies are external to firms at both stages of production, however, so there is a stable equilibrium for any magnitude of effective labor engaged in this sector. We analyze this in more detail below.

### 3.2 Intermediate-Good Technology

Each intermediate good is produced by a monopolistic competitor with cost function:

$$V(x) = v_0 + v_1x$$

where  $V(x)$  is in units of the human capital.

The total amount of human capital used by the intermediate firms must equal the amount supplied by workers:

$$MV(x) = e_I h_a N \tag{6}$$

where  $e_I$  is the effort of individuals working in the *first stage* of the conventional sector.

### 3.3 Labor Constraints

We postpone the analysis of learning, and take the amount of total work  $e_M$  to be given. The labor constraint is:

$$e_M = e_Y + e_I \tag{7}$$

At first, we take  $e_M$  as given.

## 4 Industrial Equilibrium

In both sectors, the industrial organization is similar: producers in the first stage are monopolistic competitors, while those in the second, processing, stage are perfectly competitive firms.

Let  $p$  be the price of  $x$  in units of the final good. Competition among final firms allows us to equate  $p$  to the marginal product of  $x$  in the processing stage of producing  $Y$ :

$$p = \alpha \left( \frac{e_Y h_a N}{x} \right)^{1-\alpha} \quad (8)$$

Profits for each monopolist producing an intermediate good that supplies the conventional-good sector are given by:

$$\Pi = px - w_b V(x) \quad (9)$$

where  $w_b$  is the wage for a unit of human capital, or the “base wage”. Monopolistic producers take  $w_b$  as given. Profit maximization, using (8) in (9) before taking the derivative with respect to  $x_c$ , yields the standard mark-up rule:

$$p = w_b \frac{v_1}{\alpha} \quad (10)$$

Competition among final-good firms in the C-Sector ensures that the base wage equals the marginal product of human capital:

$$w_b = \frac{\partial Y}{\partial (e_Y h_a N)} = (1 - \alpha) \left( \frac{e_Y h_a N}{x} \right)^{-\alpha} M \quad (11)$$

which relies on the symmetry of each input  $x$  in production. The variable  $h_a$ , recall, is the economy-wide average amount of human capital.

The wage  $w$  of the representative *worker* is the product the base wage  $w_b$  and the worker’s own human capital,  $h$ :

$$w = w_b h \quad (12)$$

Entrepreneurs exist to make profit. Their monopoly position allows them to do this, but only if the entry of I-firms is restricted so the number of I-firms  $M$  is relatively small. Profits vanish if there is free entry of new I-goods. We are going to assume that I-firms always make zero profit, but this is best understood as the result of a process that may take some time. Appendix

A provides more details of the process of profit and entry.

The zero-profit quantity of  $x$  produced is:<sup>2</sup>

$$x^* \equiv \frac{\alpha v_0}{(1 - \alpha)v_1} \quad (13)$$

The labor allocations at both stages of production are:

$$e_Y = (1 - \alpha) e_M \quad (14)$$

$$e_I = \alpha e_M \quad (15)$$

so that increases in  $h$  and  $N$  have no effect on effort.<sup>3</sup>

The range of I-firms is:<sup>4</sup>

$$M^* = \left( \frac{\alpha(1 - \alpha)}{v_0} \right) e_M h_a N \quad (16)$$

The wage of a worker with human capital  $h$  is given by:<sup>5</sup>

$$w = w_b h = A (e_M h_a N)^{1-\alpha} h \quad (17)$$

where  $A \equiv \frac{\alpha^{1+\alpha}(1-\alpha)^{2(1-\alpha)}}{v_0^{1-\alpha} v_1^\alpha}$ . There are increasing returns to scale through specialization – increases in both basic labor  $e_M N$  and human capital  $h$  raise the real wage. When we impose “aggregate consistency” later – the condition that  $h_a = h$  – we shall see that  $h$  affects wages more powerfully than population – not only does it enhance specialization  $M$  and raise  $w_b$ , it also makes each worker more productive and raises  $w$  proportionally.

Firms earn zero profit, so all output goes to labor in this model. Output per worker is:

$$y = \frac{Y}{N} = AN^{1-\alpha} (e_M h)^{2-\alpha} = w e_M \quad (18)$$

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<sup>2</sup>To see this, place  $w_b$  from (10) in the profit expression (9) set to zero, and solve for  $x$ .

<sup>3</sup>To derive these expressions, combine the markup (10) with the price and wage equations (8) and (11); and then use (6) to eliminate  $M$  and (7) to eliminate  $e_Y$ . Finally, insert the value for  $x^*$  from (13).

<sup>4</sup>To derive this expression, begin with (6) and substitute in (13) and (15).

<sup>5</sup>To derive (17), put (11) in (12) and then use (13) for  $x^*$  and (16) for  $M$ .

which follows from (5), symmetry of inputs, and (17). In this expression, we have imposed aggregate consistency ( $h_a = h$ ) since it reflects the well-being of the average person.

## 5 Human Capital Accumulation

Individuals accumulate personal human capital by expending effort studying.

We assume that specialization helps people learn. We let human capital be accumulated according to:

$$\dot{h} = \tilde{\delta}^\gamma h^{1-\gamma} e_L \quad (19)$$

where:

$$\tilde{\delta} \equiv \frac{M}{N} \quad (20)$$

One's own learning time  $e_L$  and one's stock of human capital  $h$  are important for generating more human capital, but there is also a positive *external* effect – represented by  $\tilde{\delta}$  – coming from the economy's specialization. Learning time is more productive when the economy is more specialized (see Goodfriend and McDermott, 1995, 1998).

We deflate the specialization term by the size of the labor force  $N = nf$  because we believe that scale exerts an offsetting effect on learning through congestion. Now substitute (16) in (20) to get:

$$\tilde{\delta} = \frac{\alpha(1-\alpha)}{v_0} e_{Ma} h_a \quad (21)$$

where the “ $a$ ” in the subscript stands for an economy-wide average over which individuals have no control. In the paper we write the accumulation equation as:

$$\dot{h} = \delta h_a^\gamma h^{1-\gamma} e_L \quad (22)$$

where:

$$\delta \equiv \left( \frac{\alpha(1-\alpha)}{v_0} e_{Ma} \right)^\gamma \quad (23)$$

Households take  $\delta$  and  $h_a$  as given when they formulate their optimal plans. Although collectively they determine  $e_{Ma}$ , each household ignores the effect that its own  $e_M$  plays in influencing  $e_{Ma}$ .

As we shall see, the presence of  $e_{Ma}$  is crucial for our theory of economic development over the very long horizon. Behavior depends on the productivity of the different activities. When  $e_{Ma}$  reaches a certain threshold, learning becomes productive enough to make it worthwhile to engage in that activity.

## 6 Growth

### 6.1 Equilibrium Conditions

The optimal policy maximizes utility from consumption:

$$\max_{c, e_j, e_l} \int_0^{\infty} N(t) u(c(t)) e^{-\rho t} dt = \max_{c, e_j, e_l} \int_0^{\infty} u(c(t)) e^{-(\rho-\eta)t} dt$$

where  $c$  is the flow of per capita consumption, and  $\rho$  is the subjective rate of discount. The second equality follows from the exogenous population growth rate  $\eta$  and the normalization of initial population at unity. There is no physical capital, so output per capita and consumption per capita are the same:  $c = y_c$ , where  $y_c \equiv \frac{Y_c}{N}$ .

The maximization is subject, first, to the effort constraint (1). Each period, work yields labor income that is spent entirely on consumption. This gives rise to the resource constraint:

$$c = y_P + w_b h e_M \tag{24}$$

Individuals believe that by learning they can increase their wage. They consider the base wages  $w_b$  to be given so that they can enhance their actual wage proportionally by acquiring human capital. Within the market sector, where they work is immaterial, so we aggregate labor into the single activity

$e_M$ .

The Hamiltonian for the household is:

$$\mathcal{H} = \ln c + \lambda \delta h_a^\gamma h^{1-\gamma} e_L + \theta_1 (y_P + w_b h e_M - c) + \theta_2 (1 - e_P - e_M - e_L)$$

where  $\lambda$  is the shadow price of  $h$  and  $\theta_1$  and  $\theta_2$  are Lagrangian multipliers for the two constraints. We assume that utility has the simple log form. Note that both  $\delta$  and  $w_b$  are positive functions of  $e_{Ma}$ , the average work effort in the urban, market sector.

The FOC's are as follows.

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \Rightarrow \frac{1}{c} = \theta_1 \quad (25)$$

$$\frac{\partial \mathcal{H}}{\partial e_P} = 0 \Rightarrow \theta_1 \frac{\partial y_P}{\partial e_P} = \theta_2 \quad (26)$$

$$\frac{\partial \mathcal{H}}{\partial e_M} = 0 \Rightarrow \theta_1 w_b h = \theta_2 \quad (27)$$

$$\frac{\partial \mathcal{H}}{\partial e_L} = 0 \Rightarrow \lambda \delta h_a^\gamma h^{1-\gamma} = \theta_2 \quad (28)$$

Two other conditions must be satisfied: the arbitrage condition:

$$\dot{\lambda} = (\rho - \eta) \lambda - \frac{\partial \mathcal{H}}{\partial h} = (\rho - \eta) \lambda - \lambda ((1 - \gamma) \delta^\gamma h_a^\gamma h^{-\gamma} e_L) - \theta_1 w_b e_M \quad (29)$$

and the transversality condition:

$$\lim_{t \rightarrow \infty} \lambda(t) h(t) e^{-(\rho - \eta)t} = 0 \quad (30)$$

## 6.2 Case 1: Primitive to Market

The first case is that in which  $e_L = 0$ . This would happen if  $\delta h_a^\gamma$  is sufficiently low that learning is not worth the effort. We derive this condition below.

In this case, there are no forward-looking dynamics. Equations (26) and (27) imply that:

$$\frac{\partial y_P}{\partial e_P} = \frac{B}{1 + e_P n} = w_b h = w \quad (31)$$

so that marginal compensation is equalized between the sectors.

The lefthand side of this equation corresponds to the upper curve in Figure 1 in the paper. The lower curve is given by (17).

As  $n$  and  $N$  grow, the top curve falls and the bottom curve rises until they meet. When they do meet, they first define a single, tangent point. We are going to discuss this point in some detail in class. It is the focus of a good part of the paper. Population growth alone plays the central role in early development and the beginning of modern growth.

We will spend a lot of time on the transition – but will do so graphically and narratively, without mathematics – and then describe what happens in the long run, when the primitive sector is gone and the economy has achieved balanced growth with just the modern technology and positive effort devoted to learning.

### 6.3 Case 2: Modern Balanced Growth

In this case, we assume that  $e_P = 0$ . The primitive sector has been abandoned because it is not productive enough once population has increased.

The first-order conditions (25), (27), (28) – with aggregate consistency  $h_a = h$  imposed – and the constraint (24) with  $e_P = 0$  yield:

$$e_M = \frac{1}{z\delta} \quad (32)$$

where  $z \equiv \lambda h$  and  $\lambda$  is the shadow price of  $h$ . It follows from the constraint (1) that:

$$e_L = 1 - \frac{1}{z\delta}$$

so that we can write the accumulation equation (19) as:

$$\dot{h} = \begin{cases} h \left( \delta - \frac{1}{z} \right) & \text{if } z \geq \frac{1}{\delta} \\ 0 & \text{if } z < \frac{1}{\delta} \end{cases} \quad (33)$$

where the inequality is the condition that determines whether or not  $e_L$  is non-negative.

For the dynamics, it is easiest to work with  $z$  and  $h$ . To get the motion equation for  $z$ , we first note that  $\frac{\dot{z}}{z} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{h}}{h}$ . Using the first-order conditions and the motion equations<sup>6</sup> allows us to express the motion of  $z$  as follows:

$$\dot{z} = \begin{cases} z[\rho - \eta + \gamma\delta] - (1 + \gamma) & \text{if } z \geq \frac{1}{\delta} \\ (\rho - \eta)z - 1 & \text{if } z < \frac{1}{\delta} \end{cases} \quad (34)$$

### 6.3.1 If $\delta$ Were Constant

In later work, I assume  $\delta$  is constant (that is, that the denominator of (20) contains  $e_{Ma}$ ). In that case, growth is straightforward. There are no transitional dynamics. The optimal policy for the representative consumer is to set  $z$  to make sure that  $\dot{z} = 0$  for all  $t$ . To do so, set (34) to zero, which yields:

$$z(t) = z_c \equiv \frac{1 + \gamma}{\rho - \eta + \gamma\delta} \quad (35)$$

This is optimal because it satisfies the transversality condition (30) as well as the other first-order conditions. The growth rate of  $h$  is constant in this dynamic equilibrium, which can be seen by putting  $z_c$  from (35) into (33). This yields:

$$g_h = \frac{\delta - (\rho - \eta)}{1 + \gamma} \quad (36)$$

which is positive if  $\delta > \rho - \eta$ , which we assume to be true.<sup>7</sup>

Real output per capita also grows at a steady rate. Output per capita is given by (18) and  $e_M = \frac{1}{z_c\delta}$ , a constant. This means that growth in  $w$  and  $y$  is the same and given from (17) as:

$$g_y = (1 - \alpha)\eta + (2 - \alpha)g_h \quad (37)$$

Population growth and human capital growth increase specialization and

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<sup>6</sup>See (33) and Equation (29).

<sup>7</sup>If there were a time-cost for children, so that  $\dot{h} = h(\delta - \frac{1}{z} - \eta)$  then in equilibrium  $g_h = \frac{\delta - \rho - \gamma\eta}{1 + \gamma}$ . Population growth has a negative effect on human capital accumulation because it raises the cost of children: effort is required to bring them up to the current average level.

raise the growth rate by  $1 - \alpha$ . There is also a proportional effect from  $h$  coming from its role in raising labor productivity and wages.

### 6.3.2 When $\delta$ Depends on $e_{Ma}$ .

In Early Development, we need learning productivity  $\delta$  to depend on work effort in the modern sector,  $e_{Ma}$ . It provides the trigger to learning, as we now explain. As people work longer hours in the urban, modern sector, it becomes easier for them to learn and raise their human capital. Growth then accelerates as  $h$  grows to raise specialization and productivity directly.

The equations above are correct before Section 6.3.1.

We still require  $z$  to be constant in Balanced Growth, so  $z(t) = z_c$  is still required. We must now solve for  $\delta$  and  $e_M$  simultaneously, calling the results  $\delta^*$  and  $e_M^*$ .

Put  $z_c$  in (35) into (32) and solve for  $\delta$ . This yields:

$$\delta = \frac{\rho - \eta}{(1 + \gamma) e_M - \gamma} \quad (38)$$

We also have a relationship between  $\delta$  and  $e_M$  from the technology itself (23). Letting  $v$  be the constant, we may re-write it succinctly as:

$$\delta = v e_M^\gamma \quad (39)$$

These two equations establish unique values  $\delta^*$  and  $e_M^*$  for any values of  $\rho$ ,  $\eta$ ,  $\gamma$ ,  $\alpha$ , and  $v_0$ . Unfortunately, we cannot solve for these in closed form. We have to use graphical and numerical methods to find them. The graph is shown as Figure 1.

The graph is very informative, however, and allows us to see that the condition for positive learning –  $e_L > 0$  so that  $e_M < 1$  — is that:

$$v > \rho - \eta \quad (40)$$

which is the same condition found above, since here  $v$  is the autonomous part of  $\delta$ .

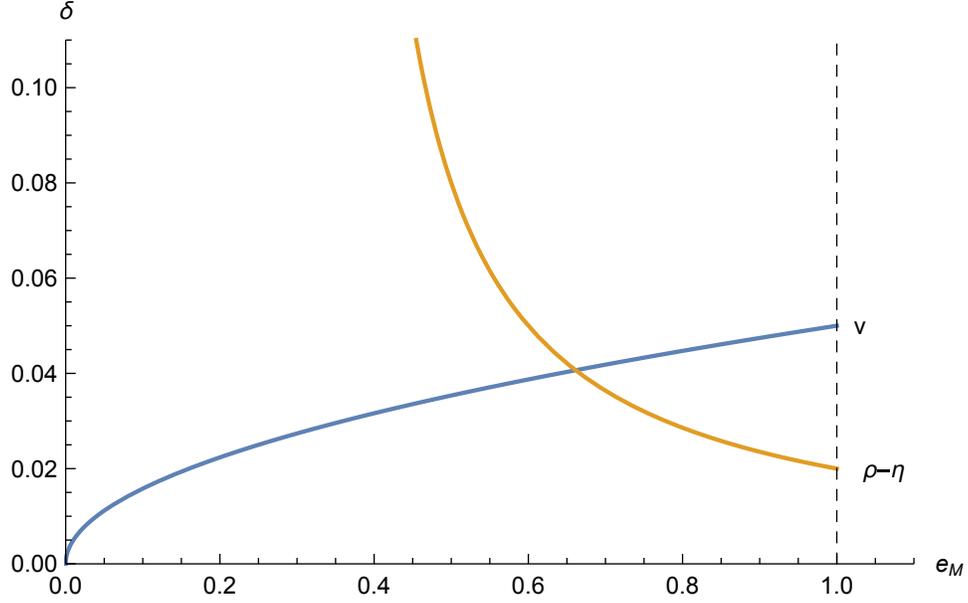


Figure 1: The Optimal  $\delta$  and  $e_M$

We can find  $z^*$  as:

$$z^* = \frac{1}{\delta^* e_M^*} \quad (41)$$

And growth in human capital is:

$$g_h^* = \delta^* - \frac{1}{z^*} \quad (42)$$

So growth in output is per person is:

$$g_y^* = (1 - \alpha) \eta + (2 - \alpha) g_h^* \quad (43)$$

## Appendix

### A Profits and Entry

Consider the solid locus labelled EQ in Figure 2. This locus shows the amount  $x_c$  that an I-firm would like to produce for the market, given the

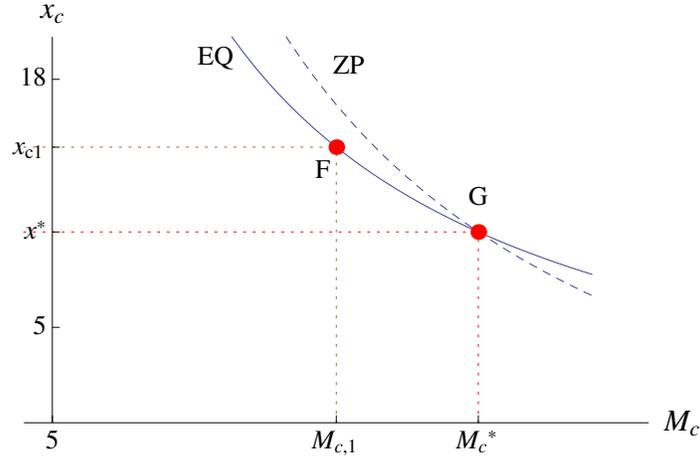


Figure 2: Monopolist Equilibrium

number of different firms  $M_c$  in the industry.<sup>8</sup> The equation of the line is:

$$x_c = \frac{\alpha^2 (e_M h N - v_0 M)}{(1 - \alpha + \alpha^2) v_1 M} \quad (44)$$

This locus corresponds to positions of equilibrium for entrepreneurs. Points here are feasible – they satisfy the labor and resource constraints – and they satisfy the markup condition that entrepreneurs believe maximizes their profit.

The other, dashed locus labeled ZP in Figure 2 corresponds to points where profit (9) is zero.<sup>9</sup> Profit is positive *below* ZP.

If the C-Sector began with a total of  $M_{c,1}$  intermediate monopoly firms – or “I-firms” – each would produce  $x_{c,1}$  at Point F and profits would be positive. If there were barriers to entry – either legal or natural – that prevented an expansion of I-firms, then this equilibrium could persist. If, however, there were no barriers,  $M_c$  would rise as new I-firms set up to

<sup>8</sup>To derive this curve, begin with the markup condition (10) and substitute in (8) and (11) for  $p_c$  and  $w_b$ . Then, eliminate  $e_{sp}$  from the equation using (7) and, finally, use the industry constraint (6) to substitute for  $e_{ic}$ .

<sup>9</sup>To derive this locus, start with (9), set to zero, and substitute, as before, for  $p_c$ ,  $w_b$ ,  $e_{sp}$  and  $e_{ic}$ . The equation is  $x_c = (\alpha E_a h N - c_0 M_c) / (c_1 M_c)$ .

produce a new intermediate good. The expansion only stops at Point G, where profits have been driven to zero. When the industry has converged to Point G, we shall say it is in “Zero-Profit Equilibrium” or ZPE.

## References

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