Early Development: Mathematics

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1 Introduction

In the paper Early Development, there are three activities. The effort devoted to each must sum to 1:

$$1 = e_P + e_M + e_L \tag{1}$$

Here, e_P is work in the primitive sector, e_M is work in the market (urban) sector, and e_L is learning effort.

2 Primitive Technology

There are f villages, each of which has n people. So the economy's population is $N = nf$. Village population grows at the rate η; the number of villages f does not change.

Total village output is:

$$Y_P = B \ln (1 + e_P n) \tag{2}$$

so that per capita output is:

$$y_p = \frac{Y_P}{n} = \frac{B}{n} \ln (1 + e_P n) \tag{3}$$

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And the marginal compensation for extra work is:

$$w_P = \frac{\partial y_P}{\partial e_P} = \frac{B}{(1 + e_P n)} \tag{4}$$

This is one of the two lines in the main figure of the paper. It is like a wage, even though there is no labor market in the primitive sector.

3 Technology and Market Structure

3.1 Final-Good Technology

Two goods are produced in a closed economy with a technology that uses a variety of specialized input goods – produced in the first stage – combined with labor augmented by human capital in the second stage. This production function is:

$$Y = \left(e_Y \bar{h} N\right)^{1-\alpha} \int_0^M (x(i))^\alpha \, di \tag{5}$$

The variable Y is total output of this sector and $e_Y \bar{h} N$ is the effective labor engaged in the second stage to produce it. The variable e_Y represents the skilled effort used to process specialized goods into the final good. N is the population and $\bar{h}$ is *average* individual human capital.[1] The variable $x(i)$ is the quantity of intermediate good i used. The limit M refers to the range of intermediate goods that are used to produce this good. We consider the first-stage production below.

Scale raises output per capita with this technology, as in Romer (1987) and Goodfriend and McDermott (1995, 1998). These scale economies are external to firms at both stages of production, however, so there is a stable

[1] Assume that human capital is distributed according to $f(h)$. That is, $f(h)$ is the number of people with human capital level h. Then $N = \int_{h_{min}}^{h_{max}} f(h)\, dh$. Aggregate human capital is given by $H = \int_{h_{min}}^{h_{max}} hf(h)\, dh$ and the average is given by $\bar{h} = \frac{H}{N}$. If everyone were alike and had human capital $\breve{h}$, which will be our assumption, then $H = \breve{h}N$. The important point is that individuals take the aggregate amount of human capital H and the average $\bar{h}$ as given when they formulate their optimal plan below. We could make the same argument about e_{sp} but there is no need, since the external effect here does not enter into the accumulation decision.

equilibrium for any magnitude of effective labor engaged in this sector. We analyze this in more detail below.

3.2 Intermediate-Good Technology

Each intermediate good is produced by a monopolistic competitor with cost function:

$$V(x) = v_0 + v_1 x$$

where $V(x)$ is in units of the human capital.

The total amount of human capital used by the intermediate firms must equal the amount supplied by workers:

$$MV(x) = e_I h N \tag{6}$$

where e_I is the effort of individuals working in the *first stage* of the conventional sector.

3.3 Labor Constraints

We postpone the analysis of learning, and take the amount of total work e_M to be given. The labor constraint is:

$$e_M = e_Y + e_I \tag{7}$$

At first, we take e_M as given.

4 Industrial Equilibrium

In both sectors, the industrial organization is similar: producers in the first stage are monopolistic competitors, while those in the second, processing, stage are perfectly competitive firms.

Let p be the price of x in units of the final good. Competition among final firms allows us to equate p to the marginal product of x in the processing stage of producing Y:

$$p = \alpha \left(\frac{e_Y \bar{h} N}{x} \right)^{1-\alpha} \tag{8}$$

Profits for each monopolist producing an intermediate good that supplies the conventional-good sector are given by:

$$\Pi = px - w_b V(x) \tag{9}$$

where w_b is the wage for a unit of human capital, or the "base wage". Monopolistic producers take w_b as given. Profit maximization, using (8) in (9) before taking the derivative with respect to x_c, yields the standard mark-up rule:

$$p = w_b \frac{v_1}{\alpha} \tag{10}$$

Competition among final-good firms in the C-Sector ensures that the base wage equals the marginal product of human capital:

$$w_b = \frac{\partial Y}{\partial \left(e_Y h N \right)} = (1 - \alpha) \left(\frac{e_Y h_a N}{x^*} \right)^{-\alpha} M \tag{11}$$

which relies on the symmetry of each input x_c in production. The variable h_a is the economy-wide average amount of human capital.

The wage w of a *worker* is the product the base wage w_b and the worker's human capital, h:

$$w = w_b h \tag{12}$$

Entrepreneurs exist to make profit. Their monopoly position allows them to do this, but only if the entry of I-firms is restricted so the number of I-firms M is relatively small. Profits vanish if there is free entry of new I-goods. We are going to assume that I-firms always make zero profit, but this is best understood as the result of a process that may take some time. Appendix A provides more details of the process of profit and entry.

The zero-profit quantity of x produced is:[2]

[2] To see this, place w_b from (10) in the profit expression (9) set to zero, and solve for x.

$$x^* \equiv \frac{\alpha v_0}{(1-\alpha)v_1} \tag{13}$$

The labor allocations at both stages of production are:

$$e_Y = (1-\alpha)\, e_M \tag{14}$$

$$e_I = \alpha e_M \tag{15}$$

so that increases in h and N have no effect on effort.[3]

The range of I-firms is:[4]

$$M^* = \left(\frac{\alpha\,(1-\alpha)}{v_0}\right) e_M h N \tag{16}$$

The wage w_s in equilibrium is given by:[5]

$$w_s = w_b h = A\,(e_M N)^{1-\alpha}\, h^{2-\alpha} \tag{17}$$

where $A \equiv \frac{\alpha^{1+\alpha}(1-\alpha)^{2(1-\alpha)}}{v_0^{1-\alpha} v_1^{\alpha}}$. There are increasing returns to scale through specialization – increases in both basic labor $e_M N$ and human capital h raise the real wage. The latter affects wages and per capita output most strongly – not only does it enhance specialization, it also makes each worker more productive.

We can express output per person as:

$$y = \frac{Y}{N} = AN^{1-\alpha}\,(e_M h)^{2-\alpha} = w_s e_M \tag{18}$$

which follows from (5), symmetry of inputs, and (17).

[3] To derive these expressions, combine the markup (10) with the price and wage equations (8) and (11); and then use (6) to eliminate M and (7) to eliminate e_Y. Finally, insert the value for x^* from (13).

[4] To derive this expression, begin with (6) and substitute in (13) and (15).

[5] To derive (17), put (11) in (12) and then use (13) for x^* and (16) for M_c.

5 Human Capital Accumulation

Individuals accumulate personal human capital by expending effort studying.

We assume that specialization helps people learn. We let human capital be accumulated according to:

$$\dot{h} = \tilde{\delta}^{\gamma} h^{1-\gamma} e_L \tag{19}$$

where:

$$\tilde{\delta} \equiv \frac{M}{N} \tag{20}$$

One's own learning time e_L and one's stock of human capital h are important for generating more human capital, but there is also a positive effect – represented by $\tilde{\delta}$ – coming from the economy's specialization. Learning time is more productive when the economy is more specialized (see Goodfriend and McDermott, 1995, 1998).

We deflate the specialization term by the size of the labor force employed in the market sector $N = nf$ because we believe that scale exerts an offsetting effect on learning through congestion. Now substitute (16) in (20) to get:

$$\tilde{\delta} = \frac{\alpha (1-\alpha)}{v_0} e_{ma} h_a \tag{21}$$

where the "a" in the subscript stands for an economy-wide average over which individuals have no control. In the paper we write the accumulation equation as:

$$\dot{h} = \delta h_a^{\gamma} h^{1-\gamma} e_L \tag{22}$$

where:

$$\delta \equiv \left(\frac{\alpha (1-\alpha)}{v_0} e_{Ma} \right)^{\gamma} \tag{23}$$

Households take δ and h_a as given when they formulate their optimal plans. Although collectively they determine e_{Ma}, each household ignores the effect that its own e_M plays in influencing e_{Ma}.

6 Growth

6.1 Equilibrium Conditions

The optimal policy maximizes utility from consumption:

$$\max_{c,e_j,e_l} \int_0^\infty N(t)\, u(c(t))\, e^{-\rho t} dt = \max_{c,e_j,e_l} \int_0^\infty u(c(t))\, e^{-(\rho-\eta)t} dt$$

where c is the flow of per capita consumption, and ρ is the subjective rate of discount. The second equality follows from the exogenous population growth rate η and the normalization of initial population at unity. There is no physical capital, so output per capita and consumption per capita are the same: $c = y_c$, where $y_c \equiv \frac{Y_c}{N}$.

The maximization is subject, first, to the effort constraint (1). Each period, work yields labor income that is spent entirely on consumption. This gives rise to the resource constraint:

$$c = y_P + w_b h e_M \tag{24}$$

Individuals believe that by learning they can increase their wage. They consider the base wages w_b to be given so that they can enhance their actual wage proportionally by acquiring human capital. Within the market sector, where they work is immaterial, so we aggregate labor into the single activity e_M.

The Hamiltonian for the household is:

$$\mathcal{H} = \ln c + \lambda\, \delta h_a^{\gamma} h^{1-\gamma} e_L + \theta_1 (y_P + w_b h e_M - c) + \theta_2 (1 - e_P - e_M - e_L)$$

where λ is the shadow price of h and θ_1 and θ_2 are Lagrangian multipliers for the two constraints. We assume that utility has the simple log form. Note that both δ and w_b are positive functions of e_{Ma}, the average work effort in the urban, market sector.

The FOC's are as follows.

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \;\Rightarrow\; \frac{1}{c} = \theta_1 \tag{25}$$

$$\frac{\partial \mathcal{H}}{\partial e_P} = 0 \;\Rightarrow\; \theta_1 \frac{\partial y_P}{\partial e_P} = \theta_2 \tag{26}$$

$$\frac{\partial \mathcal{H}}{\partial e_M} = 0 \;\Rightarrow\; \theta_1 w_b h = \theta_2 \tag{27}$$

$$\frac{\partial \mathcal{H}}{\partial e_L} = 0 \;\Rightarrow\; \lambda \delta h_a^{\gamma} h^{1-\gamma} = \theta_2 \tag{28}$$

Two other conditions must be satisfied: the arbitrage condition:

$$\dot{\lambda} = (\rho - \eta)\,\lambda - \frac{\partial \mathcal{H}}{\partial h} = (\rho - \eta)\,\lambda - \lambda \left((1-\gamma)\,\delta^{\gamma} h_a^{\gamma} h^{-\gamma} e_L - \eta \right) - \theta_1 w_b e_M \tag{29}$$

and the transversality condition:

$$\lim_{t \to \infty} \lambda(t)\, h(t)\, e^{-(\rho-\eta)t} = 0 \tag{30}$$

6.2 Case 1: Primitive to Market

The first case is that in which $e_L = 0$. This would happen if δh_a^{γ} is sufficiently low that learning is not worth the effort. We derive this condition below.

In this case, their are no forward looking dynamics. Equations (26) and (27) imply that:

$$\frac{\partial y_P}{\partial e_p} = \frac{B}{1 + e_P n} = w_b h = w \tag{31}$$

so that marginal compensation is equalized between the sectors.

The lefthand side of this equation corresponds to the upper curve in Figure 1 in the paper. The lower curve is given by (17).

As $n$and N grow, the top curve falls and the bottom curve rises until they meet.

6.3 Case 2: Modern Balanced Growth

In this case, we assume that $e_P = 0$. The primitive sector has been abandoned because it is not productive enough once population has increased.

The first-order conditions (25), (27), (28) – with aggregate consistency $h_a = h$ imposed – and the constraint (24) with $e_P = 0$ yield:

$$e_M = \frac{1}{z\delta} \tag{32}$$

where $z \equiv \lambda h$ and λ is the shadow price of h. It follows from the constraint (1) that:

$$e_L = 1 - \frac{1}{z\delta}$$

so that we can write the accumulation equation (19) as:

$$\dot{h} = \begin{cases} h\left(\delta - \frac{1}{z}\right) & if\ z \geq \frac{1}{\delta} \\ 0 & if\ z < \frac{1}{\delta} \end{cases} \tag{33}$$

where the inequality is the condition that determines whether or not e_L is non-negative.

For the dynamics, it is easiest to work with z and h. To get the motion equation for z, we first note that $\frac{\dot{z}}{z} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{h}}{h}$. Using the first-order conditions and the motion equations[6] allows us to express the motion of z as follows:

$$\dot{z} = \begin{cases} z[\rho - \eta + \gamma\delta] - (1+\gamma) & if\ z \geq \frac{1}{A} \\ (\rho - \eta)\, z - 1 & if\ z < \frac{1}{A} \end{cases} \tag{34}$$

6.3.1 If δ Were Constant

In later work, I assume δ is constant (that is, that the denominator of (20) contains e_{Ma}). In that case, growth is straightforward. There are no transitional dynamics. The optimal policy for the representative consumer is to set $z$to make sure that $\dot{z} = 0$ for all t. To do so, set (34) to zero, which

[6] See (33) and Equation (**??**)in Appendix **??**.

yields:

$$z(t) = z_c \equiv \frac{1+\gamma}{\rho - \eta + \gamma\delta} \tag{35}$$

This is optimal because it satisfies the transversality condition (30) as well as the other first-order conditions. The growth rate of h is constant in this dynamic equilibrium, which can be seen by putting z_c from (35) into (33). This yields:

$$g_h = \frac{\delta - (\rho - \eta)}{1+\gamma} \tag{36}$$

which is positive if $\delta > \rho - \eta$, which we assume to be true.[7]

Real output per capita also grows at a steady rate. Output per capita is given by (18) and $e_M = \frac{1}{z_c\delta}$, a constant. This means that growth in w and y is the same and given from (17) as:

$$g_y = (1-\alpha)\,\eta + (2-\alpha)\,g_h \tag{37}$$

Population growth and human capital growth increase specialization and raise the growth rate by $1-\alpha$. There is also a proportional effect from h coming from its role in raising labor productivity and wages.

6.3.2 When δ Depends on e_{Ma}.

In Early Development, we need learning productivity δ to depend on work effort in the modern sector, e_{Ma}. It provides the trigger to learning, as we now explain. Moreover, it makes sense that there is an additional effect besides specialization. As people work longer hours in the urban, modern sector, it becomes easier for them to learn and raise their human capital.

The equations above are correct before Section 6.3.1.

The method now is to use (32) to eliminate z in (34). Then set (34) to zero: it is still necessary that z be constant to satisfy transversality. this

[7] If there were a time-cost for children, so that $\dot{h} = h\left(\delta - \frac{1}{z} - \eta\right)$ then in equilibrium $g_h = \frac{\delta-\rho-\gamma\eta}{1+\gamma}$. Population growth has a negative effect on human capital accumulation because it raises the cost of children: effort is required to bring them up to the current average level.

yields:

$$\delta = \frac{\rho - \eta}{(1+\gamma)\,e_M - \gamma} \tag{38}$$

We also have a relationship between δ and e_M from the technology itself (23). Letting v be the constant, we may re-write it succinctly as:

$$\delta = v e_M^{\gamma} \tag{39}$$

These two equations establish unique values δ^* and e_M^* for any values of ρ, η, γ, α, and v_0. Unfortunately, we cannot solve for these in closed form. We have to use numerical methods to find them.

The graph is very informative, however, and allows us to see that the condition for positive learning – $e_L > 0$ so that $e_M < 1$ — is that:

$$v > \rho - \eta \tag{40}$$

which is the same condition found above, since here v is the autonomous part of δ.

We can find z^* as:

$$z^* = \frac{1}{\delta^* e_M^*} \tag{41}$$

And growth in human capital is:

$$g_h^* = \delta^* - \frac{1}{z^*} \tag{42}$$

So growth in output is per person is:

$$g_y^* = (1-\alpha)\,\eta + (2-\alpha)\,g_h^* \tag{43}$$

Appendix

A Profits and Entry

Consider the solid locus labelled EQ in Figure 1. This locus shows the amount x_c that an I-firm would like to produce for the market, given the

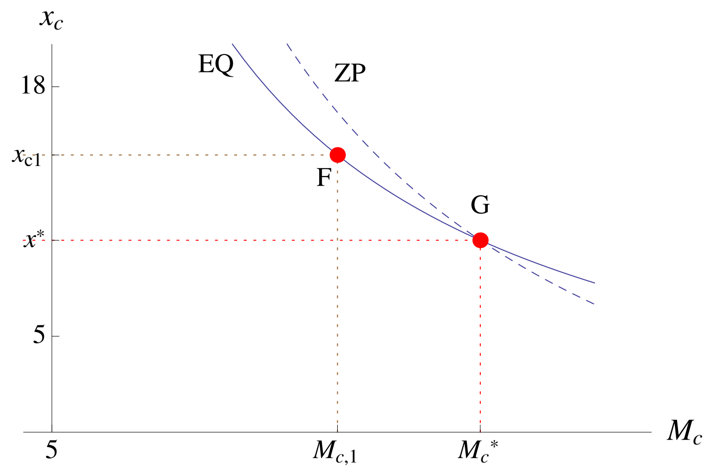

Figure 1: Monopolist Equilibrium

number of different firms M_c in the industry.[8] The equation of the line is:

$$x_c = \frac{\alpha^2 \left(e_M hN - v_0 M\right)}{\left(1 - \alpha + \alpha^2\right) v_1 M} \tag{44}$$

This locus corresponds to positions of equilibrium for entrepreneurs. Points here are feasible – they satisfy the labor and resource constraints – and they satisfy the markup condition that entrepreneurs believe maximizes their profit.

The other, dashed locus labeled ZP in Figure 1 corresponds to points where profit (9) is zero.[9] Profit is positive *below* ZP.

If the C-Sector began with a total of M_{c1} intermediate monopoly firms – or "I-firms" – each would produce x_{c1} at Point F and profits would be positive. If there were barriers to entry – either legal or natural – that prevented an expansion of I-firms, then this equilibrium could persist. If, however, there were no barriers, M_c would rise as new I-firms set up to

[8]To derive this curve, begin with the markup condition (10) and substitute in (8) and (11) for p_c and w_b. Then, eliminate e_{sp} from the equation using (7) and, finally, use the industry constraint (6) to substitute for e_{ic}.

[9]To derive this locus, start with (9), set to zero, and substitute, as before, for p_c, w_b, e_{sp} and e_{ic}. The equation is $x_c = \left(\alpha E_a hN - c_0 M_c\right) / \left(c_1 M_c\right)$.

produce a new intermediate good. The expansion only stops at Point G, where profits have been driven to zero. When the industry has converged to Point G, we shall say it is in "Zero-Profit Equilibrium" or ZPE.

References

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