Time Aggregation and the Relationship between Inflation and Money Growth

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Abstract

Using panel data for 99 countries, we confirm that the measured elasticity of prices with respect to money is higher, and closer to unity, the higher is money growth and the longer the time horizon over which the data are averaged. We propose two contexts within which to explain this result. In one, the true model of inflation involves a lagged response to money growth. In the other, there is negative correlation between shocks to inflation and money growth. Our empirical results can be explained if high-money-growth countries have (1) shorter lags or (2) less negative correlation, when compared to countries with low money growth.

JEL Codes: E31, E47, E52,C32

Keywords: inflation, money growth, monetary policy, time averaging

1 Introduction

Studies of the relationship between inflation and money growth are numerous and vary by historical period, country sample, and method. To one extent or another, all test the quantity theory of money – that is, whether the elasticity of prices with

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respect to money is unity. One common approach is to use cross-country, long-run, time-averaged data in a regression of inflation on money growth and real output growth. This produces estimated coefficients near unity only if the sample contains countries with high money growth. Moreover, as the time horizon over which the data are averaged falls, the estimated coefficient also falls, and in a pronounced manner for countries with low money growth.[1] Our purpose in this paper is twofold: to corroborate these empirical 'facts' and to explain this pattern of *estimated* coefficients even if the *true* coefficient is unity.

Using a panel of 99 countries with data sampled at annual, half-decade and decade intervals, as well as at the cross-sectional level, we confirm that the money growth-inflation relationship is different for low-money-growth countries than for high-money-growth countries. We also find that greater time-averaging of the data increases the estimate from a conventional regression of inflation on money growth and output growth. For high-money-growth countries, the effect of time-averaging on the relationship between money growth and inflation flattens out quickly as time-averaging increases, but this is not true for low-money-growth countries. We propose two explanations for this empirical pattern. Both are worked out in the context of money growth that is positively serially correlated, an assumption that appears to have empirical support.

Our first explanation is based on a model in which inflation is related to a distributed lag in money growth. To get the differential effects between high- and low-money-growth countries that we observe, it must be the case that the inflation responses of high-money-growth countries are associated with a shorter lag structure than in low-money-growth countries. When the true model contains only one lag of money growth, we demonstrate that the probability limit of the estimated coefficient using averaged data is equal to the true long-run elasticity as the averaging interval

[1] The impact of temporal aggregation on empirical estimates has been studied for a long time dating back to Moriguchi (1970) and Zellner and Montmarquette (1971). In these papers, the temporal aggregation problems addressed arise when a researcher has data sampled at a longer frequency than appropriate for the underlying theory. We have temporal aggregation problems in reverse – the data are sampled at a finer frequency than has been thought appropriate for estimating the theoretical long run relationship between inflation and money.

gets very large. However, if the true model has more than one lag, this is not true; the estimated relationship will always fall short of the true relationship.

Our second explanation is based on a model in which inflation shocks are offset by the monetary authority. A Taylor Rule or explicit inflation targeting, for example, would introduce negative correlation between the errors in money growth and inflation. Alternatively, if shocks to money demand are offset by the monetary authority, or there is measurement error that affects money growth and inflation inversely, then negative correlation between shocks to money growth and inflation may be present. In either case, this type of correlation induces negative bias in the estimated elasticity of inflation with respect to money growth. We show that time-averaging mitigates the bias as the averaging horizon increases, if money growth is positively serially correlated. Because low-money-growth countries may be more likely to fit this second explanation, we should expect that the estimated relationship between money growth and inflation will be smaller for low-money-growth countries than for high-money-growth countries.

There is a large body of empirical work that has yielded empirical findings that are consistent with our own findings and the explanations we offer. The long-run relationship between inflation and money growth has been investigated by Vogel (1974), Barro (1990), Duck (1993), McCandless and Weber (1995), Gerlach (1995), and Lucas (1996) using a cross-section of countries. This work generally supports the coefficient of unity on money growth, but if the sample is reduced by taking away the countries with the highest money growth, the coefficient falls and the confidence interval no longer contains one.

Single-country studies like Lucas (1980) find the quantity theory of money holds for the United States when using “filtered” data. Sargent and Surico (2011) find that the unitary elasticity hypothesis breaks down in the United States when monetary rules are introduced. Using a distributed lag model, McCallum and Nelson (2010) find qualified support for the unitary elasticity of money growth for the United States. They also test proportionality for the G-7 countries and find little support. They argue that if the true process that generates inflation involves lags of money growth, averaging may not allow the researcher to accurately uncover the full effect of money – a claim that we investigate.

Panel studies have also been applied to test the quantity theory of money. Teles and Uhlig (2013) conjecture that widespread inflation targeting, whether tacit or not and beginning as early as 1990, was responsible for the breakdown in the money growth-inflation relationship among OECD countries after 1990. Dwyer and Fisher (2009) report that the correlation between money growth and inflation is close to unity for high inflation countries, but that the correlation between money growth and inflation declines at lower rates of money growth, a finding that echoes Gerlach (1995). They also claim that when countries target inflation and the target is serially correlated, the correlation between money growth and inflation rises with time aggregation. Frain (2004) examines the unitary elasticity hypothesis by focusing on high- and low-inflation countries. He uses recursive regression and does not reject it for either group. DeGrauwe and Polan (2005) investigate the effect of both increasing time aggregation and increasing magnitudes of money growth for a panel of countries. They find virtually no relationship between money growth and inflation when money growth is low.

The paper is organized as follows. In Section 2 we discuss the sources and construction of the data. Section 3 contains our long-run results using the country-average cross-section data for our two sub-samples of low-money-growth countries and high-money-growth countries. Section 4 shows our panel results at three levels of time aggregation – decade, half-decade, and annual. Section 5 lays out the model structure, using the simplest, counterfactual case. Section 6 analyzes a model based on differences in lag structure that is consistent with the different empirical results for low- and high-money-growth countries. Section 7 offers an alternative explanation that is based on negative correlation between money growth and inflation. Section 8 concludes.

2 Data

Our data on money and prices come from the International Monetary Fund, *International Financial Statistics* (CD-ROM, 2014).[2] We use data sampled annually. The price level is measured by the consumer price index (CPI). We measure annual inflation – Δp – using the first-difference in the natural log of the CPI. For most countries, the time series coverage of the CPI is consistent and widely-reported, although the initial year varies widely. The money data are more problematic: the definition of money is not consistent across countries, and the IFS reports two versions of broad money, M2 and MQM (money plus quasi-money). For those countries that report both M2 and MQM, we select one or the other using an algorithm that balances length of series and its latest date. We select M2 if it ends at least 3 years *later* than MQM *and* has no more than 10 *fewer* observations than MQM. Otherwise, we choose MQM. Whichever series we choose, we call it "M2".[3] We create our measure of annual money growth – Δm – by taking the first-difference of the natural log of M2.

We also collect data on real output for our study. We use data from the Penn World Table v.8.0 which reports three distinct output series, available annually. We use the national-accounts based series $RGDP^{NA}$, which is similar to real output in earlier versions of the PWT data. We measure annual output growth – Δq – by taking the first-difference of the natural log of $RGDP^{NA}$.

We trim the data so that they are *rectangular by country* in the three variables – Δp, Δm, and Δq – that enter our regressions below. That is, for each country, we take the largest possible data available for each country for which all three series start and end on the same dates. We think it is important that the averaged data cover the same time period for each variable in a given country. For some countries, we have data from 1951-2011; for others, the start date may be as late as the early

[2]Data for the United Kingdom's CPI prior to 1988 is from the Office of National Statistics.

[3]Our two measures of M2 are broadly comparable and where countries report both, the correlation between the series is 0.69. Neither series, however, is available for as many countries as is the CPI data. In general, MQM is a longer series than M2 but typically ends in 2008. While M2 coverage is shorter (having a later start year), it generally includes the year 2010 and therefore has the advantage of including the financial crisis period.

1990s. However, for inclusion in our data set, we require that each country have at least 20 years of data.[4] We are left with a sample of 99 countries.

3 Long Run, Cross-Section Results

Our first exercise is to examine the relationship between money growth and inflation using annualized, long-run, cross-country averages. This relationship is shown in Figure 1. The straight line has a slope of one and passes through the sample averages of Δm_j (0.179) and Δp_j (0.121), where the subscript indexes the country. The relationship appears to be very strong: the points lie close to the line. Three countries have money growth and inflation rates that are far above the mean of the annualized averages. These countries are Argentina (ARG), the Democratic Republic of Congo (formerly Zaire (ZAR)), and Brazil (BRA).

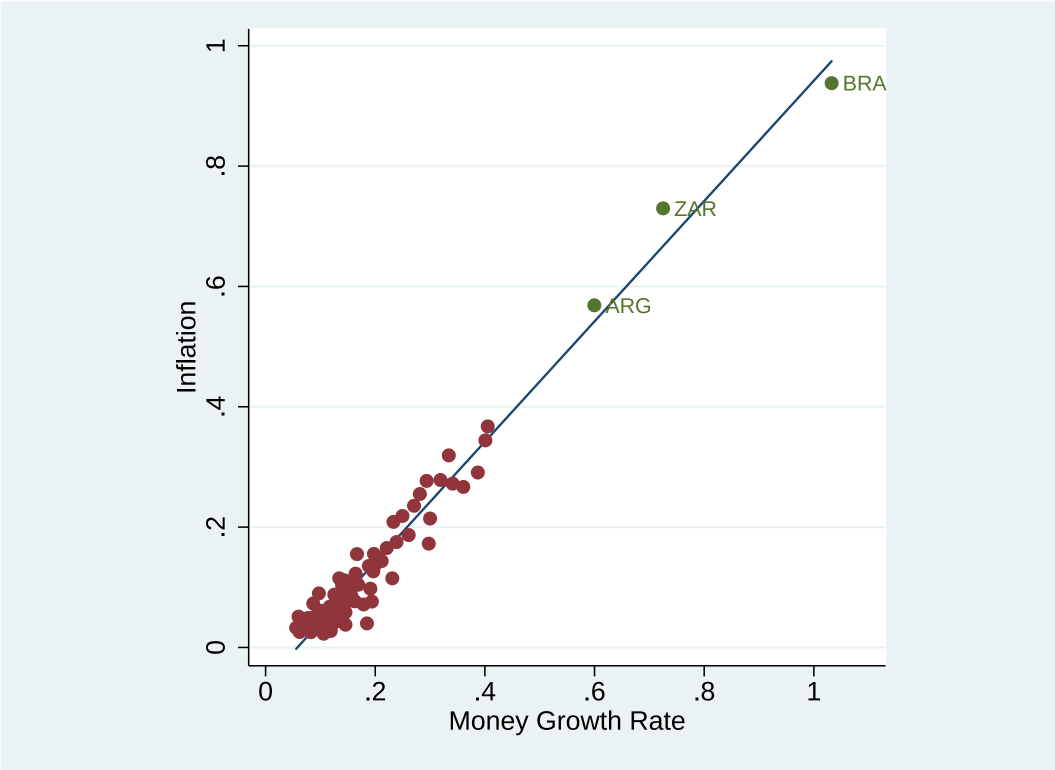

Figure 1: Inflation and Money Growth by Country

[4] We also drop Ecuador. Irregularities in the money data made Ecuador an extreme outlier: the difference between inflation and money growth was four standard deviations from the sample cross-country mean.

A striking feature of Figure 1 is the cloud of countries with low rates of money growth. To see what is going on in this cloud, we divide our countries into two groups. In the first are countries with long-run average M2 money growth rates less than 15 percent per year; that is, with $\Delta m_j < 0.15$. We call these 58 countries the "*Low-Money-Growth Countries*" or LMG countries. We call the remaining 41 countries with $\Delta m_j \geq 0.15$ the "*High-Money-Growth Countries*" or HMG countries. This division does not change throughout the paper. We chose the cut-off of 15 percent because it is a round number close to the mean (0.179) and the median (0.137) of the long run country-average annualized money growth rates.[5]

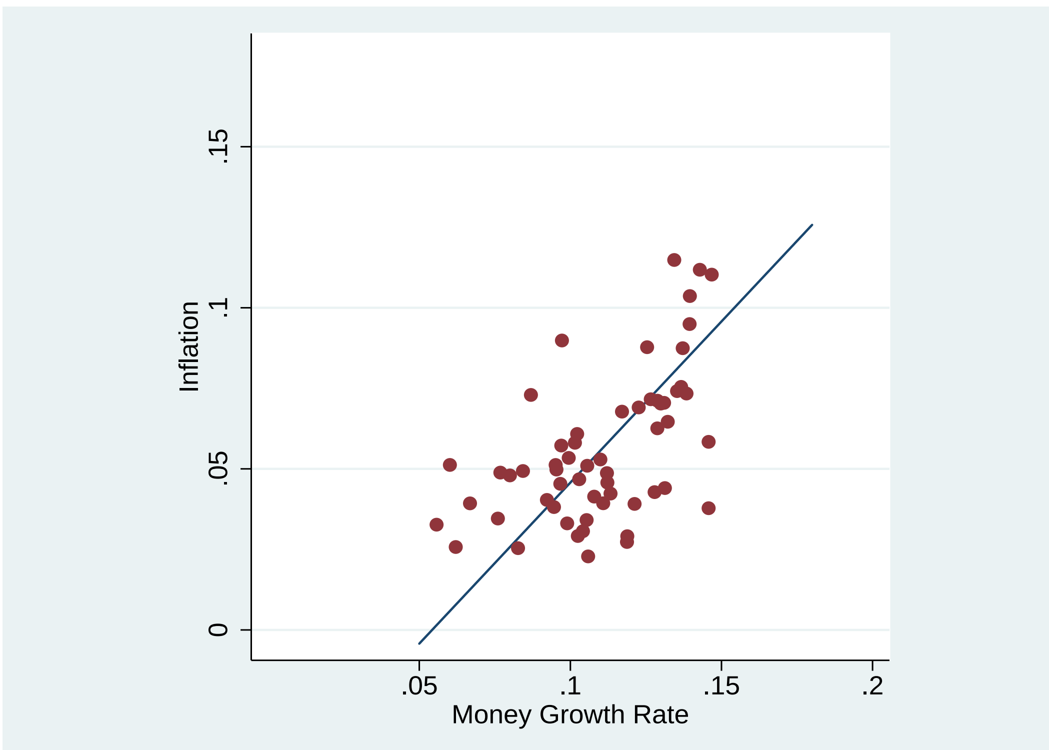

Figure 2: Inflation and Money Growth: LMG Countries

The cross-sectional relationship between money growth and inflation for the low-money-growth (LMG) countries is shown in Figure 2. The points in this figure do *not* cluster neatly along the straight line with slope of one. Figures 1 and 2 show when the HMG countries are excluded, the relationship between money growth and inflation is not only less precise, but also may be less than unitary in magnitude.

[5]We also used a 20 percent cutoff. The results here and in later Sections were little changed.

Table 1: Inflation and Money Growth: Country Averages

Country Averages				
$\Delta p_j = \alpha + \beta \Delta m_j + \delta \Delta q_j + \varepsilon_j$				
Sample	$\hat{\beta}$	*95% Confidence Interval*	*N*	R^2
All	0.97***	0.93 - 1.02	99	0.98
LMG	0.79***	0.60 - 0.98	58	0.63
HMG	0.97***	0.91 - 1.04	41	0.98
Note:***Significant at 0.01; ** at 0.05; * at 0.10.				

In these figures, we have not accounted for real output growth. It may be that its absence has a bigger impact on the estimated money growth-inflation relationship for LMG countries because real output growth and money growth are similar in magnitude. By contrast, in HMG countries, the effect of real output growth may be marginalized by the effect of money growth.

We test whether this is the case, using the following the long-run, cross-section regression equation:

$$\Delta p_j = \alpha + \beta \Delta m_j + \delta \Delta q_j + \varepsilon_j, \quad j = 1 ... N \tag{1}$$

where ε_j is assumed to be an iid random variable with mean 0 and variance σ^2, and N is the number of countries in the sample. We do not restrict the coefficients on either money growth Δm or real output growth Δq.

We estimate Equation (1) for our sample of 99 countries and the sub-samples of LMG and HMG countries. Our focus will be on the magnitude of $\hat{\beta}$, and how close it is to 1.0. The results are shown in Table 1.[6] Row 1 presents the results using the full sample of all 99 countries. The estimate $\hat{\beta}$ is very close to 1.0 and the 95 percent confidence interval contains 1.0. Moreover, 98 percent of the variation in inflation across countries is explained by money growth and output growth.

In the second row of Table 1, we present the results with the sample restricted to the LMG countries. The point estimate $\hat{\beta}$ falls to 0.79. For this sample of countries,

[6] We also ran all of our results using the narrow measure of money M1. The estimates of β are a bit lower, but the pattern of results is similar.

the 95 percent confidence interval no longer contains 1.0. Contrast this with the results in Row 3 for the HMG countries. For these countries, $\hat{\beta} = 0.97$ and the 95 percent confidence interval contains 1.0. A t-test of the difference between the $\hat{\beta}$'s for the high- and low-money-growth groups delivers a p-value of 0.03.[7]

Another way to gauge the effect of money growth on the estimated β is to conduct recursive regressions. We order our cross-section of countries according to their average money growth – from low to high – and then run a series of regressions beginning with the first 20 countries, those with the lowest average money growth. Each subsequent regression adds one country – the one with the next highest average money growth – until we have incorporated all 99 countries. In Figure 3 we show the estimated β's and the 95 percent confidence boundary that result. Along the x-axis, we show the rate of money growth that corresponds to the added country. The vertical line at 0.15 separates the LMG group from the HMG group: 58 countries reside to the left of the boundary, and 41 countries to the right. Like Gerlach (1995), Frain (2004), and Dwyer and Fisher (2009), we find that $\hat{\beta}$ rises and the confidence bounds shrink as countries with higher money growth are added to the sample. Further, we see that $\hat{\beta}$ remains well below 1.0 at money growth rates less than about 30 percent.

To illustrate how the sample over which we run the regression matters, we also run *reverse* recursive regressions. The result of this exercise is shown in Figure 4. Now we assemble the countries in order of *decreasing* money growth. The first sample includes the 20 countries with the *highest* average money growth over their history. Subsequent regressions add one country at a time. The x-axis registers the next lowest of the highest-money-growth countries added to the regression. The confidence interval contains 1.0 in all cases, but the point estimate falls when countries with money growth less than 15 percent are added to the sample. We see that the point estimate only declines below 1.0 when countries with money growth at 10 percent or lower are added.

[7]The p-value refers to the test on the coefficient of the interaction term with money growth where we augmented (1) with a dummy variable for low money growth. The dummy was added to the intercept and interacted with both money growth and real output growth.

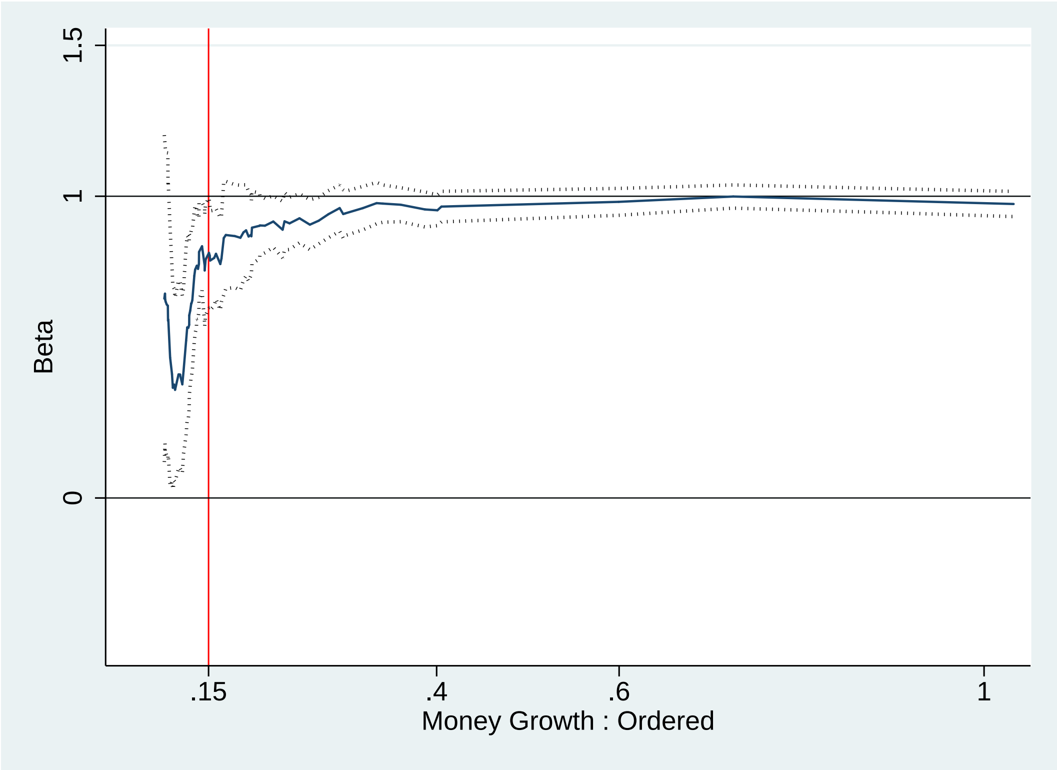

Figure 3: Recursive Regression Betas

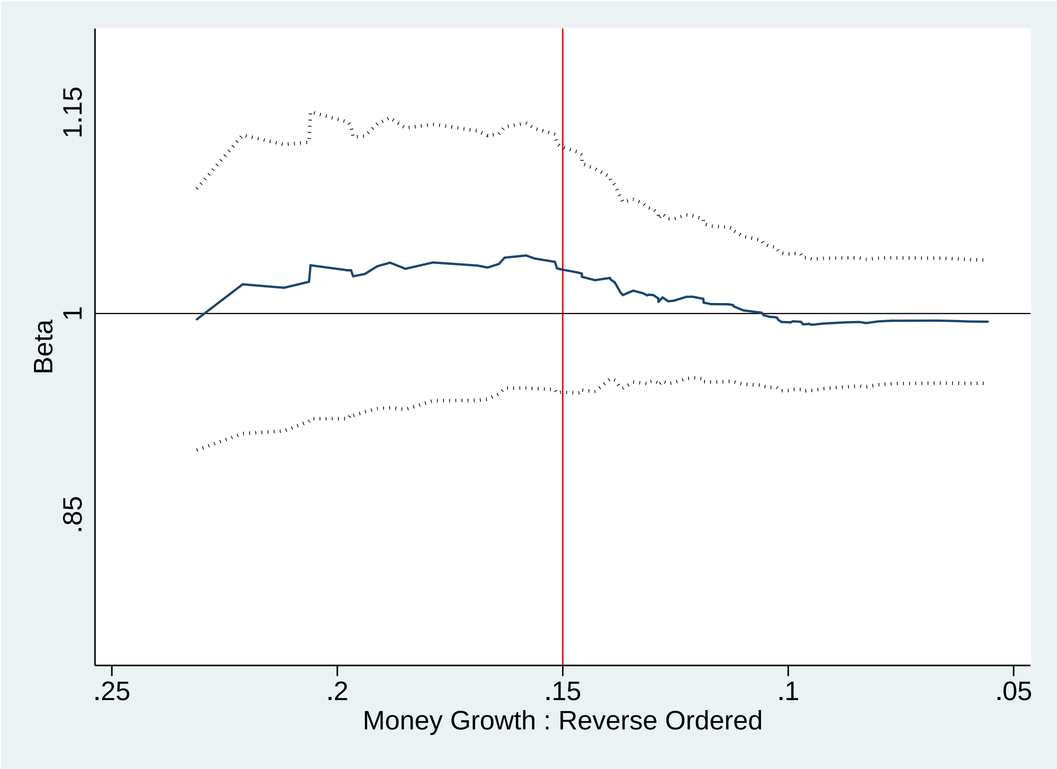

Figure 4: Reverse Recursive Regression Betas

Taken as whole, the results in Table 1 and Figures 1 – 4 suggest that the evidence in favor of a unit elasticity of prices with respect to money depends on the influence of countries with high money growth. As we will see, this interpretation continues to hold when we move from using country cross-section data to panel data.

4 Results for Different Time Aggregation

We now investigate the effect of averaging the data ("time aggregation") at different time horizons K using panel data. We construct three panels beginning with calendar decade averages ($K = 10$), then proceed to calendar half-decade averages ($K = 5$), and ending with annual data ($K = 1$). At each level, we consider three different samples: all countries, the LMG countries, and the HMG countries.

Our estimating equation is:

$$\Delta p_{j,t} = \alpha + \beta \Delta m_{j,t} + \delta \Delta q_{j,t} + \alpha_j + \lambda_t + \varepsilon_{j,t} \tag{2}$$

where j indexes the country, t indexes either decade, half-decade or year, α_j are the country fixed-effects, and δ_t are a complete set of time effects corresponding to the level of time aggregation. Fixed effects estimation corrects for the omission of country-specific, time-invariant, unobservable factors that might be correlated with $\Delta m_{j,t}$ and bias our estimates of β. We use time effects because there may also be time-specific events - like an oil price shock - that affect each country's inflation rate similarly.

In constructing the decade and half-decade averages for each country, we require that at least 80 percent of the *annual* observations over the relevant period be available. If they are not, we exclude these periods from the estimation.

The panel regression results are shown in Table 2. For decade and half-decade data, the full panel results – see the rows labeled "*All*" in Table 2 – show that the confidence interval around $\hat{\beta}$ includes 1.0. For annual data, we estimate $\hat{\beta} = 0.80$, and the confidence interval extends to 0.92.

Table 2: Inflation and Money Growth: Panel Results

$$\Delta p_{j,t} = \alpha + \beta \Delta m_{j,t} + \delta \Delta q_{j,t} + \alpha_j + \lambda_t + \varepsilon_{j,t}$$

Decade Results ($K = 10$)				
Sample	$\hat{\beta}$	*95% Confidence Interval*	*Obs* (N)	R^2
All	0.96***	0.91-1.01	413 (99)	0.95
LMG	0.40**	0.28 - 0.52	241 (58)	0.55
HMG	0.97***	0.92 - 1.02	172 (41)	0.96
Half-Decade Results ($K = 5$)				
Sample	$\hat{\beta}$	*95% Confidence Interval*	*Obs* (N)	R^2
All	0.95***	0.90-1.00	850 (99)	0.94
LMG	0.29**	0.20 - 0.38	497 (58)	0.47
HMG	0.96***	0.92 - 1.02	353 (41)	0.95
Annual Results ($K = 1$)				
Sample	$\hat{\beta}$	*95% Confidence Interval*	*Obs* (N)	R^2
All	0.80***	0.68-0.92	4371 (99)	0.77
LMG	0.13**	0.09 - 0.17	2562 (58)	0.33
HMG	0.85***	0.74 - 0.96	1809 (41)	0.80

NOTE: The number in parentheses is the number of countries N in the panel.

Standard errors clustered by country. Fixed Effects and time dummies included.

Note:***Significant at 0.01; ** at 0.05; * at 0.10.

The more interesting results concern the difference between the LMG and HMG samples. First, the quantity theory prediction is rejected for the LMG countries across all three panels, but for the HMG countries it is not rejected for the decade and half-decade panels. Further, for the HMG countries, $\hat{\beta}$ is much higher with the annual data (0.80) than for the LMG countries (0.13). Second, time aggregation matters for the estimate of β more for the LMG countries than for the HMG set. For LMG nations, as the degree of time aggregation K rises from one year to five years to ten years, $\hat{\beta}$ increases from 0.13 to 0.29 to 0.40. In no case, moreover, does the confidence interval contain 1.0, nor does it contain 0. For the HMG sample, on the other hand, $\hat{\beta}$ rises as we move from using annual data to half-decade-averaged data or from annual data to decade-averaged data. However, $\hat{\beta}$ changes very little as we transition from half-decade-averaged data to decade-averaged data.

5 A Simple Model of Inflation

Money appears to have a unit elastic effect on prices only - that is β is unity - when money growth is high and when the data are aggregated over a long horizon. In this section, to fix ideas and introduce notation, we consider a very simple model of inflation. We assume a true model that features no lags in the inflation-money growth relationship. We offer this model as a counterfactual to our results; this case does *not* deliver the effects we find in the data.

In the interest of tractability and generality, going forward we use the following notation: $y \equiv \Delta p$ and $x \equiv \Delta m$. The true state of nature for any country is now written:

$$y_t = \alpha + \beta_0 x_t + \epsilon_t \tag{3}$$

where t indexes time by year. In this simple model, β_0 is the long-run elasticity; if the quantity theory prediction is correct, then $\beta_0 = 1.0$. Our results in Table 2 suggest that this is possible, and we will continue to focus on that possibility. At this point, we assume that money growth x_t is not serially correlated nor correlated with the error ϵ_t at any lag – assumptions we will change below.

A regression using annual data – if (3) is true – will produce a consistent estimate of β_0. It is well-known that the probability limit of the OLS estimator of β_0 is the true value:

$$p\lim \hat{\beta}_0 = \beta_0 + \frac{\sigma_{x\epsilon}}{\sigma_x^2} = \beta_0$$

where $\sigma_{x\epsilon}$ and σ_x^2 are, respectively, the unobserved population covariance and variance, and $\sigma_{x\epsilon} = 0$ by assumption.

Now we consider running the simple regression using averaged data. Let $Y_{K,t}$, $X_{K,t}$, and $\Lambda_{K,t}$ be the K-period, non-overlapping averages of, respectively, y, x, and

ϵ that end in year t. That is, the average for x is:

$$X_{K,t} = \frac{1}{K} \sum_{i=0}^{K-1} x_{t-i} \tag{4}$$

and $Y_{K,t}$ and $\Lambda_{K,t}$ are defined analogously. Notice that we have defined K so that if $K = 1$, the data are *not* averaged.

Our interest from now on will center on the coefficient estimate - we call it β - from the following regression using K-averaged data:

$$Y_{K,t} = \gamma + \beta X_{K,t} + \Lambda_{K,t} \tag{5}$$

Now take the average of both sides of the *true model* in (3) to obtain:

$$Y_{K,t} = \alpha + \beta_0 X_{K,t} + \Lambda_{K,t} \tag{6}$$

The OLS estimator of β in (5) is :

$$\hat{\beta} = \frac{\text{cov}\,(Y, X)}{\text{var}\,(X)} = \frac{\text{cov}\,(\alpha + \beta_0 X + \Lambda, X)}{\text{var}\,(X)} \tag{7}$$

where "cov" and "var" are, respectively, the sampling covariance and variance. The second equality in (7) uses the true value for Y from equation (6). Now take the probability limit of $\hat{\beta}$ in (7) to get:

$$p \lim \hat{\beta} = \beta_0 + \frac{\sigma_{X\Lambda}}{\sigma_X^2} = \beta_0 \tag{8}$$

where $\sigma_{X\Lambda} = 0$. This holds true when there is no correlation between x and ϵ, either contemporaneously or at any lag.

The important observation that comes out of this simple model is that averaging the data should *not* affect the estimated coefficient in (5) — no matter whether $K = 1$ or $K > 1$. This conclusion, however, does *not* fit the pattern of results across Tables 1 – 2: we see that the estimates of β across different degrees of K-averaging are quite different, especially for the LMG countries.

Next, we provide two explanations for the pattern of results we observe.

6 First Explanation: Variable Lags

Our first explanation alters the true model from the previous section by incorporating lags of x_t. This approach is most closely identified with the work of McCallum and Nelson (2010). The true model is now:

$$y_t = \theta + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \ldots + \beta_H x_{t-H} + \epsilon_t \tag{9}$$

Equation (9) says that inflation is determined by a distributed lag of money growth with $H > 0$ lags.

In a model with lags, the long-run elasticity of money with respect to prices is:

$$B = \sum_{i=0}^{H} \beta_i \tag{10}$$

We interpret the quantity theory of money to mean that $B = 1.0$.

We assume that money growth x_t is an autocorrelated AR(1) process:

$$x_t = \phi + \rho x_{t-1} + \omega_t \tag{11}$$

where $0 < \rho < 1$. Further, we assume that there is no correlation between the errors in (9) and (11) so that:

$$\sigma_{\epsilon_t \omega_t} = \sigma_{x_{t-j} \epsilon_t} = 0 \;\; j = 0, ..., H \tag{12}$$

Using variations of this model, we can provide an explanation for the pattern of coefficient estimates across Tables 1 – 2 for the HMG and LMG countries.

6.1 Case 1: Short Lags, H = 1

We start with a short lag model letting $H = 1$ in (9): only current and one lag of money growth are relevant to the generation of inflation. The long-run elasticity is now $B = \beta_0 + \beta_1$ from (10). This lag structure may be more likely to characterize HMG countries where firms raise prices quickly, within a year or two, after they have come to expect that money will continue to grow at a high rate. We show that the $H = 1$ case is the *only* case that can explain the pattern of empirical results for the HMG countries, under the assumption that the quantity theory holds in the long run.

Now, suppose that we do not know the true model and estimate (5) using K-averaged data. In this case, when the true model is given by (9) – (12) and $H = 1$, the estimator of the coefficient on money growth in (5) is given by:

$$\hat{\beta} = \frac{\operatorname{cov}(Y, X)}{\operatorname{var}(X)} = \frac{\operatorname{cov}(\alpha + \beta_0 X + \beta_1 X_{-1} + \Lambda, X)}{\operatorname{var}(X)} \tag{13}$$

where X_{-1} is $X_{K,t}$ with each element x lagged one year. Taking the probability limit gives:

$$\begin{aligned} p\lim\hat{\beta} &= \beta_0 + \beta_1 \frac{\sigma_{XX_{-1}}}{\sigma_X^2} + \frac{\sigma_{X\Lambda}}{\sigma_X^2} \\ &= \beta_0 + \beta_1 \frac{\sigma_{XX_{-1}}}{\sigma_X^2} \end{aligned} \tag{14}$$

We find that two of the terms are the same as in (8), but we have an additional term that is proportional to β_1. As before, the last term drops out because the error is uncorrelated with the elements of X.

In Appendices A.1 and A.2 we derive the expressions for σ_X^2 and $\sigma_{XX_{-1}}$ under the maintained autocorrelation in (11). There, we show that their ratio can be expressed as:

$$\frac{\sigma_{XX_{-1}}}{\sigma_X^2} \equiv C(K, \rho) = \frac{\left(\rho K + \left(\rho + \frac{1}{\rho}\right) M(K, \rho)\right)}{(K + 2M(K, \rho))} \tag{15}$$

where:

$$M\left(K,\rho\right)\equiv\sum_{i=1}^{K-1}\rho^{i}\left(K-i\right)=\frac{\rho\left(K-1-\rho K+\rho^{K}\right)}{\left(\rho-1\right)^{2}} \tag{16}$$

The right-hand side of (16) is a well-known closed form solution for the summation in (16).

Combine (14) and (15) to see that:

$$p\lim\hat{\beta}=\beta_{0}+C\left(K,\rho\right)\beta_{1} \tag{17}$$

Here, we see that the probability limit of $\hat{\beta}$ will not equal the long-run elasticity $B=\beta_0+\beta_1$ unless $C\left(K,\rho\right)=1$.

We now focus on the $C\left(K,\rho\right)$ function which is a non-linear function of K and ρ. Table 3 show values for $C(K,\rho)$ for various values of K and ρ. Figure 5 graphs $C(K,\rho)$ for $K=1$ to $K=20$, and for two values of ρ: $\rho=.50$ (solid circles) and $\rho=.25$ (hollow circles).

Table 3 and Figure 5 demonstrate the following properties of $C(K,\rho)$:

1. $0\leq C\left(K,\rho\right)<1$
2. $C_K>0$ and $C_\rho>0$
3. $\lim_{K\to\infty}C\left(K,\rho\right)=1.0$
4. $\lim_{\rho\to 1}C\left(K,\rho\right)=1.0$
5. $C\left(1,\rho\right)=\rho$

The function $C(K,\rho)$ is less than one, rises as K or ρ increase, and approaches 1.0 as either K goes to infinity or ρ goes to 1.0.

Table 3 shows the effect of averaging at longer intervals: as the averaging interval $K\to\infty$ – which practically speaking occurs with 50 observations – $C(K,\rho)$ gets very close to 1.0. Thus, β_1 receives nearly full weight in (17) and the expected estimate of $\hat{\beta}$, in the limit, captures the true long-run elasticity.

Table 3: $C(K, \rho)$ Function Values

	$C(K,\rho)$					
$\frac{\rho\rightarrow}{K\downarrow}$	0	0.1	0.25	0.5	0.75	0.99
1	0.000	0.100	0.250	0.500	0.750	0.990
5	0.800	0.829	0.866	0.913	0.954	0.998
10	0.900	0.916	0.937	0.962	0.980	0.999
50	0.980	0.984	0.988	0.993	0.997	1.000

Note: cell entries calculated using equations (15) and (16).

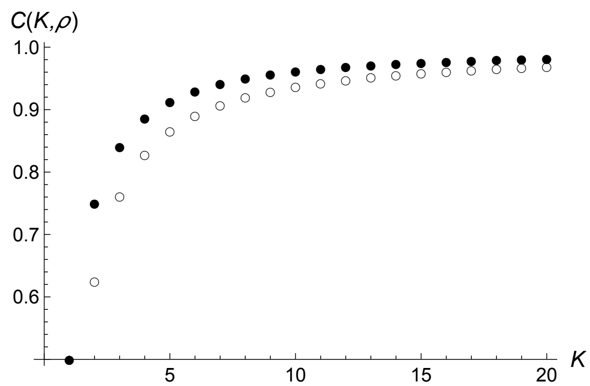

Figure 5: The $C\left(K,\rho\right)$ Function

In essence, long time-averaging produces a 'pseudo-correction' to the estimate $\hat{\beta}$ from (5).[8] However, in the case of $K = 1$ (i.e. using annual data), this pseudo-correction does *not* arise when the true model contains only one (autocorrelated) lag of x_t: the $p\lim\hat{\beta}$ would equal the familiar omitted variables bias formula in the presence of autocorrelated regressors: $\beta_0 + \rho\beta_1$. Indeed, Table 3 shows that $C(K,\rho) > \rho$ for any $K > 1$. This result establishes that time-averaging moves the estimate $\hat{\beta}$ closer to the true long run elasticity.

This important result helps explain the pattern of estimates of β across Tables 1 and 2. Recall that in Section 5 where $H = 0$, we showed that a model with no lags cannot explain the pattern of estimates of β as K increases for the HMG countries. If H were really zero, averaging the data over larger intervals K would have *no* effect on $\hat{\beta}$. This is not what we observe: we see that $\hat{\beta} = 0.80$ when $K = 1$ and $\hat{\beta} = 0.97$

[8]Note that when $\rho = 0$, $C\left(K,0\right) = 1 - \frac{1}{K}$ so that as $K \rightarrow \infty$, $C\left(K,0\right)$ approaches 1, just as in the case of a positive ρ.

with the long run cross-section data.

By contrast, we see that if $H = 1$, then Table 3 shows (assuming $\rho = .5$) that the weight on β_1 in (17) goes from 0.50 to 0.913 to 0.962, as we increase K from 1 to 5 to 10. This pattern fits our results in Table 2 where we see the biggest change occur between $K = 1$ and $K = 5$.

The case of $H = 1$ is central because averaged data *in the limit* delivers the correct long-run elasticity B. This will *not* be true if the true lag length is longer, as we show next.

6.2 Long Lags: H > 1

Here, we consider a longer lag structure in x_t. This lag structure may be more suitable for LMG countries than for HMG countries. In countries with low money growth, changes to money growth may not be immediately detectable by agents and there can be considerable difficulty in separating aggregate changes in demand from relative changes. Prices will adjust slowly and may not fully reflect the change in money for several years.

In this case, if we estimate (5) when the true model is (9) with $H > 1$, the estimator is:

$$\hat{\beta} = \frac{\text{cov}(Y, X)}{\text{var}(X)} = \frac{\text{cov}(\alpha + \beta_0 X + \beta_1 X_{-1} + \beta_2 X_{-2} + \beta_3 X_{-3} ... + \beta_H X_{-H} + \Lambda, X)}{\text{var}(X)} \tag{18}$$

It can be shown, using the technique of Appendix A.2, that the covariance of any lag $j > 1$ of X and X itself is:

$$\sigma_{XX-j} = \rho^{j-1} \sigma_{XX_{-1}} \tag{19}$$

This means that we can now express the probability limit of the estimator from (5) with data averaged over interval K as:

$$p\lim \hat{\beta} \quad = \quad \beta_0 + C(K, \rho)\beta_1 + \rho C(K, \rho)\beta_2 + \ldots \rho^{H-1} C(K, \rho)\beta_H$$

$$= \quad \beta_0 + C\left(K,\rho\right)\sum_{i=1}^{H}\beta_i\rho^{i-1} \tag{20}$$

where the function $C(K,\rho)$ is the same as derived in Section 6.1. As before, the $C(K,\rho)$ function depends only on K and ρ; it does not depend on the lag length H in the true model, nor does it vary with each lag $i = 1, ..., H$ in (20).

We can see that:

$$p\lim\hat{\beta} = \beta_0 + C\left(K,\rho\right)\sum_{i=1}^{H}\beta_i\rho^{i-1} < \beta_0 + \sum_{i=1}^{H}\beta_i \equiv B$$

Recall that in the short lag case where $H = 1$, $p\lim\hat{\beta}$ approached the long-run elasticity B as $K \to \infty$. Here, we see that when the true model has more than one lag, the probability limit must be strictly *less* than B even when $K \to \infty$. Time averaging moves the estimate of β closer to B, but it will never reach it. To the extent that ρ is small and the size of the true lag coefficients are small – the difference $\left(B - p\lim\hat{\beta}\right)$ could be sizable. On the other hand, if ρ is small and the true lag coefficients decay rapidly with $\beta_1 \gg \beta_{i-1}$, then $p\lim\hat{\beta} \approx \beta_0 + C\left(K,\rho\right)\beta_1$.

We can use Equation (20) to help explain two features of the pattern of LMG coefficient estimates in Table 2. First, the smallest LMG coefficient in Table 2 is 0.13 for annual data ($K = 1$), so we can infer from (20) that β_0 is *smaller* than that – if we are justified in identifying the estimated $\hat{\beta}$ with the $p\lim\hat{\beta}$. If the quantity theory is even approximately correct, the small $\hat{\beta}$ for annual data suggests that H is large so that $\sum_{i=1}^{H}\beta_i$ is large. That is, the inference that β_0 is quite small suggests that if the quantity theory is true, then β_i $(i > 0)$ sum to $(1 - \beta_0)$, and they are numerous and individually small in magnitude.[9]

Second, the fact that an increase in the length of K-averaging pushes the expected estimate of $\hat{\beta}$ close to B without ever reaching it can explain why the LMG coefficients in Table 2 go from 0.13 to 0.29 to 0.40 as K rises from 1 to 5 to 10. We illustrate with a simple, but general, example. Assume that $H = 10$ and that

[9] Alternatively, it may also be that β_1 is large and close to $(1 - \beta_0)$ with $\sum_{i=2}^{H}\beta_i$ very small. In this latter case, the model is approximated by the $H = 1$ case.

$\beta_0 = 0.10$. Further, assume that each of the 10 lag coefficients are *the same*: $\beta_i = \bar{\beta} = \frac{1-0.10}{10} = 0.09$ (for $0 < i \leq H$). These numerical values satisfy the quantity theory in the long run. Now use (20) to calculate $p\lim\hat{\beta}$ for various values of K and ρ. The results from this exercise are shown in Table 4. We label these contrived estimates $p\,lim\hat{\beta}'_{K,\rho}$.

The cells of Table 4 show what happens to $p\lim\hat{\beta}'$ (using the contrived value for $\beta_i = \bar{\beta}$) when the averaging horizon is K years long and serial correlation of x_t is ρ. It is clear from Table 4 that, even though we set up the example so that the long-run elasticity of money to prices is 1.0, $p\lim\hat{\beta}'$ is far below 1.0 for any (K,ρ) pair. It does, however, rise with both K and ρ.

Table 4: Calculated Prob Limits for $\hat{\beta}'$

	$p\,lim\hat{\beta}'_{K,\rho}$					
$\frac{\rho\rightarrow}{K\downarrow}$	0	0.1	0.25	0.5	0.75	.99
1	0.100	0.110	0.130	0.190	0.355	0.952
5	0.172	0.183	0.204	0.264	0.424	0.959
10	0.181	0.192	0.212	0.273	0.433	0.960
∞	0.190	0.200	0.220	0.280	0.440	0.961

Calculated using equations (20) and (15).

Assumes $H = 10$ lags and $\beta_0 = 0.10$ and $\beta_i = 0.09 = \bar{\beta}$, i=1,...,10.

Consider the two extreme values $p\,lim\hat{\beta}'_{\infty,\rho}$ and $p\,lim\hat{\beta}'_{1,\rho}$. These are the values reported in the last and first rows of Table 4. When the true lag coefficients in (9) are the same and equal to some value $\bar{\beta}$, as in our example, we can show that the *difference* of these probability limits is given by:

$$p\,lim\hat{\beta}'_{\infty,\rho} - p\,lim\hat{\beta}'_{1,\rho} = \bar{\beta}\left(1-\rho^{H-1}\right) \simeq \bar{\beta} \tag{21}$$

In our example, $\bar{\beta} = .09$. This is very close to the difference between the values in the last and first row in Table 4.

Equation (21) is derived using Properties 3 and 5 of the $C(K,\rho)$ function introduced in Section 6.1. For the annual case, since $C(1,\rho) = \rho$, we apply (20) to see that:

$$p\,lim\hat{\beta}'_{1,\rho} = \beta_0 + \rho\beta_1 + \rho^2\beta_2 + \rho^3\beta_3 + ... + \rho^H\beta_H \tag{22}$$

On the other hand, the $\lim_{K\to\infty}C\,(K,\rho) = 1$, so that:

$$p\,lim\hat{\beta}'_{\infty,\rho} = \beta_0 + \beta_1 + \rho\beta_2 + \rho^2\beta_3 + ... + \rho^{H-1}\beta_H \tag{23}$$

If we set $\beta_i = \bar{\beta}$ (for $i > 0$) and subtract (22) from (23), we can simplify to get (21).[10]

The result in (21) puts bounds on what we can reasonably expect from time averaging of the data. At the most, time averaging can be expected to raise the estimated coefficient in (5) by the amount $\bar{\beta}$, which we may think of in general as an approximation to the *mean of the true coefficients* in (9) at lag 1 and beyond. In our simple example, it is exactly that mean. If $H = 3$ or $H = 4$, $\bar{\beta}$ will be larger than the value of .09 that we have assumed in our example with $H = 10$.

To sum up, our empirical results are generally consistent with a model in which the true inflation generating process is a distributed lag in money growth. In particular, it can explain why time averaging can move the estimated coefficient β into the confidence interval containing 1.0 for HMG countries because of their short lag structure, but not for LMG countries that are likely to have a longer lag structure. The key insight is contained in (21). For HMG countries, using time-averaged data allows the researcher to recover almost all of the influence of past money growth that is missing when only annual data is used. Why? Because there is only one lag to account for, so $\bar{\beta}$ captures it all. This is not so with LMG countries: $\bar{\beta}$ captures only a fraction $\frac{1}{H}$ of the influence of past money growth.

[10] When $\rho = 0$, $C(K,\rho) = 1 - \frac{1}{K}$ and as $K \to \infty$, $C(K,\rho) \to 1$. Therefore (21) still holds.

7 Second Explanation: Negative Correlation

We have so far maintained the assumption that money growth shocks are random and independent of those to inflation. This may be true in HMG countries, but is not likely to characterize LMG countries. Here, we allow for negative correlation between shocks to money growth and inflation. For tractability, we return to the assumption that money growth affects inflation only contemporaneously so the true model is (3) and β_0 is the long-run elasticity. We continue to assume positive autocorrelation in x as in (11). Negative correlation between shocks to inflation and money growth, ϵ_t and ω_t, could arise in a few contexts: inflation targeting, financial or other innovations; or measurement error.[11] A policy of inflation targeting in which shocks to inflation are offset by countervailing shocks to money growth will induce such negative correlation. It is also possible that financial and technological innovations – which are more likely to occur in LMG countries which tend to be more developed – induce negative correlation between shocks to money growth and inflation. Thus, in this section, we now assume that $\sigma_{\epsilon\omega} = \sigma_{x\epsilon} < 0$ in LMG countries.

When there is negative correlation between the errors, there is also negative correlation between the averages X and Λ, $\sigma_{X\Lambda} < 0$. The negative correlation will be complicated by the fact that each x in X is autocorrelated. In this case, if we estimate (5), we have:

$$p\text{lim}\hat{\beta} = \beta_0 + \frac{\sigma_{X\Lambda}}{\sigma_X^2} < \beta_0 \tag{24}$$

The expected estimate is *smaller* than the true long-run elasticity β_0, which is equal to 1.0 if the quantity theory is correct.

We have found the expression for σ_X^2 already. In Appendix A.3 we derive an expression for $\sigma_{X\Lambda}$ and show that the ratio is given by:

$$\frac{\sigma_{X\Lambda}}{\sigma_{\text{X}}^2} = \frac{\sigma_{x\epsilon}}{\sigma_x^2} G(K, \rho) < 0 \tag{25}$$

[11] We find support in the data for this assumption. We calculated values for $\sigma_{\epsilon\omega}$ using OLS residuals from (3) and (11) for all 99 countries. For the LMG group, 64 percent were negative and the average $\sigma_{\epsilon\omega}$ was -0.08. For the HMG group, in contrast, only 36 percent were negative and the average was 0.20.

where:

$$G(K,\rho) \equiv \left(\frac{K + M(K,\rho)}{K + 2M(K,\rho)}\right) \tag{26}$$

and the $M(K,\rho)$ function is given by (16). Combine (24) and (25) to get:

$$p\lim\hat{\beta} = \beta_0 + \frac{\sigma_{x\epsilon}}{\sigma_x^2} G(K,\rho) < \beta_0 \tag{27}$$

We call $G(K,\rho)$ the *asymptotic bias proportion function*: it is the proportion of the asymptotic bias in annual data that remains when estimating using data that is averaged over K years. As we will see, the benefit of estimating (5) with K-averaged data is that $p\lim\hat{\beta}$ gets closer to the true β_0 as K rises.

Figure 6 plots $G(K,\rho)$ in (26) as a function of K, for two values of ρ, $\rho = .50$ (solid circles) and $\rho = .25$ (hollow circles). We can show that:

1. $\frac{1}{1+\rho} < G(K,\rho) \leq 1$
2. $G_K < 0$ and $G_\rho < 0$
3. $G(K,0) = G(1,\rho) = 1$
4. $\lim_{K\to\infty} G(K,\rho) = \frac{1}{1+\rho}$

For the purpose of explaining our results, the key property of (26) is that $G_K < 0$, so that $p\lim\hat{\beta}$ gets closer to the true β_0 as K rises. That is, the asymptotic bias can be *reduced* by increasing the time averaging horizon. This effect, however, is limited, as we see from properties 1 and 4. Even in the limit as $K \to \infty$, $G(K,\rho)$ can go no lower than $\frac{1}{1+\rho}$.

This explanation can also account for our pattern of estimated coefficients in Table 2 for the LMG countries. Take a simple example. If $\beta_0 = 1$ and we take our point estimate of $\hat{\beta} = 0.13$ for the LMG countries at annual frequency in Table 2 as the value of $p\lim\hat{\beta}$, then we can use (27) to find an estimate for the ratio: $\frac{\sigma_{x\epsilon}}{\sigma_x^2} = -.87$. We can find this easily because the frequency of the data is annual and $G(1,\rho) = 1$. Now use (27) at *decade* frequency. Table 2 tells us the estimated coefficient is 0.40. As before, identify 0.40 with the probability limit then, using

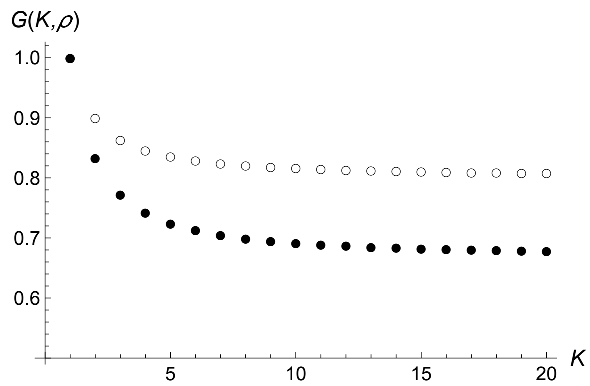

Figure 6: bias proportion and K

numerical methods, take (27) with $K = 10$ and $\frac{\sigma_{x\epsilon}}{\sigma_x^2} = -0.87$ to find $\rho = 0.51$. This may be a reasonable value for the serial correlation in money growth. We do not wish to push the calibration exercise too far, but only to show that this explanation may be consistent with our results for LMG countries.

Our explanation for the difference in results for HMG countries and LMG countries in this case is based on the difference in σ_{xe}, the negative correlation between inflation and money growth. It is reasonable to assert that LMG countries are more likely to fit this proposed structure. HMG countries are not likely to have negative feedback from inflation to money growth – if anything, that feedback could be positive – so averaging would not yield an estimate of β closer to its true value.[12]

8 Conclusion

It is commonplace to say that money affects the price level proportionally. It can be difficult, however, to find proportionality with data averaged over any time interval when money growth is on average less than 15 percent per year - a result we corroborate using a panel of 99 countries with data sampled at the annual, half-decade, decade, and cross-section frequency. We have offered two explanations for why this might be the case, even if the true long-run elasticity of money with respect to prices is unity.

[12]Where a positive simultaneity bias exists, the bias will be upward but will still be mitigated when data are averaged at increasing levels of time aggregation, provided the regressor has positive serial correlation.

One explanation relies on a distributed lag of money growth in the true process that generates inflation. In this model, the true long-run elasticity is the sum of the coefficients on money growth, both contemporaneous and lagged. We were able to show that the greater the number of lags in the true model, the greater the *difference* between the true long-run elasticity – which in the quantity theory is 1.0 – and the expected coefficient from a regression of time-averaged inflation on time-averaged money growth. It is a short step from there to reason that countries with more lags in the true generating equation will therefore do worse in approximating the true long-run elasticity using time-averaged data. This story is compatible with results for our low-money-growth countries. On the other hand, when there is a short pass-through of money growth to inflation, using time-averaged data is more likely to capture the underlying true model. This story is consistent with our results for the high-money-growth countries.

Our second explanation relies on the existence of negative correlation between shocks to inflation and shocks to money growth. The negative correlation creates downward bias in the estimate of the coefficient relating money growth to inflation. Using time-averaged data moves the probability limit of the coefficient estimate closer to its true value – if money growth is positively serially correlated – but at most by a half. If negative correlation is strongest in low-money-growth countries – perhaps because the monetary authority is more likely to target inflation or offset money demand shocks – then these countries will have lower coefficient estimates when using averaged data compared to high-money-growth countries. However, a greater degree of averaging of the data mitigates the downward bias and moves the estimated relationship closer to the true relationship. This explanation provides an alternative interpretation to the empirical results for the low-money-growth countries.

A common approach to testing hypotheses about the long-run elasticity of money with respect to prices is to estimate a model of inflation regressed on money growth using data that are time-averaged. However, these results are likely to be misleading because using time-averaged data produces a coefficient that is below the true long-run elasticity for either of the two reasons described above. For high-money-growth countries where inflation is likely to be generated by a short lag in money growth, the discrepancy between the estimated relationship and the true relationship – in the limit of averaging – is nil. Alternatively, because high-money-growth countries are not likely to demonstrate negative feedback from inflation to money growth, they avoid the influence of negative bias, for any degree of time-averaging. For low-money-growth countries, however, we demonstrated that using time averaged data to estimate the money growth-inflation relationship will mitigate any downward bias.

Given potentially different explanations for the low money growth country results, it is difficult to know whether differences in the estimated money growth-inflation relationship between high- and low-money-growth countries arise because long averaging is not able to capture – in a statistical sense – the full impact of money growth on inflation, or because low-money-growth countries are fundamentally different from countries with high money growth.

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A Variance and Covariance Expressions

A.1 The Variance of X

Here we derive the variance of X_K in (4), which is the denominator of (15).

In general, for a linear combination of random variables $W = a_1x_1 + a_2x_2 ... a_Kx_K$, and in which the x variables are correlated, the formula for the variance of W is:

$$Var\left(W\right) = \sum_{i=1}^{K} a_i^2 Var\left(x_i\right) + \sum_{i>j}^{K} 2a_ia_jCov\left(x_i, x_j\right) \tag{28}$$

The first term accounts for each of the K contemporaneous elements x_i and the second for any covariance that may exist between x_i and x_j. Now let $W = X_{K,t} \equiv X$ as defined by (4) in the text.

For X, it is useful to construct a matrix like that in Table 5, which applies to the case of $K = 5$. The first row and column list each term of X in (4), where $a \equiv \frac{1}{K}$. The expressions in each inner cell are the covariances between the terms. The variance of X is simply the sum of the contents of these inner cells. Thus, the first term in (28) corresponds to the sum of the elements along the main diagonal. Since there are K identical terms, and using the definition of a, we may write it as:

$$K\left(\frac{1}{K^2}\right)\sigma_x^2$$

Since x_t is an AR(1) series, the covariance between any two distinct terms of Z can be written:

$$Cov\left(x_t, x_{t-i}\right) = Cov\left(x_{t-i}, x_t\right) = \rho^i\sigma_x^2$$

where i is the time distance between terms. This means that the matrix is symmetric, so we only have to add the terms above the main diagonal, and then double that value. Each term has the value $a^2\sigma_x^2 = \left(\frac{1}{K^2}\right)\sigma_x^2$ in common. They only differ in the power of ρ and the number of identical terms associated with each power. The pattern is easily seen in Table 5: the number of terms in ρ^i is $K - i$ where i is the distance backward from t. Thus, the second term in (28) can be written as:

$$2\left(\frac{1}{K^2}\right)\sigma_x^2 \sum_{i=1}^{K-1} \rho^i \left(K - i\right) \tag{29}$$

Table 5: Variance Terms of Z: $K = 5$

	ax_t	ax_{t-1}	ax_{t-2}	ax_{t-3}	ax_{t-4}
ax_t	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho^3\sigma_x^2$	$a^2\rho^4\sigma_x^2$
ax_{t-1}	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho^3\sigma_x^2$
ax_{t-2}	$a^2\rho^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$
ax_{t-3}	$a^2\rho^3\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$
ax_{t-4}	$a^2\rho^4\sigma_x^2$	$a^2\rho^3\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$

The expression for the variance of X is then:

$$\sigma_X^2 = \frac{\sigma_x^2}{K^2}\left(K + 2\sum_{i=1}^{K-1}\rho^i\left(K-i\right)\right) = \frac{\sigma_x^2}{K^2}\left(K + 2M\left(K,\rho\right)\right) \tag{30}$$

where $M\left(K,\rho\right) \equiv \sum_{i=1}^{K-1}\rho^i\left(K-i\right)$ and is expressed in closed form in (16) in the text.

Notice, too, that since x_t is the AR(1) process shown in (11), we can express its variance by:

$$\sigma_x^2 = \frac{\sigma_\omega^2}{1-\rho^2} \tag{31}$$

In the text, we use σ_x^2, but we can express this variance in terms of the underlying variance of the error ω_t, as well as the value of ρ.

Finally, if $\rho = 0$, then we see from Table 5 that the variance of X is only the first term in (30) : $\sigma_X^2 = \frac{\sigma_x^2}{K}$.

A.2 The Covariance of X and X_{-1}

We proceed as above, first forming a 5×5 helper matrix ($K = 5$) whose elements are the covariances of individual terms. The first column of Table 6 is a list of the terms of $X_{5,t}$ while the first row contains the elements in $X_{5,t-1}$. The covariance we seek is the sum of all the entries in the inner cells.

We note that each term has the common factor $a^2\sigma_x^2 = \frac{\sigma_x^2}{K^2}$. Ignoring this factor for now, along the main diagonal we have K terms of ρ. Along other diagonals, the pattern emerges: there are $K-i$ terms in both ρ^{i+1} and ρ^{i-1} . This allows us to

Table 6: Covariance Terms of X and X_{-1}: $K = 5$

	ax_{t-1}	ax_{t-2}	ax_{t-3}	ax_{t-4}	ax_{t-5}
ax_t	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho^3\sigma_x^2$	$a^2\rho^4\sigma_x^2$	$a^2\rho^5\sigma_x^2$
ax_{t-1}	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho^3\sigma_x^2$	$a^2\rho^4\sigma_x^2$
ax_{t-2}	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho^3\sigma_x^2$
ax_{t-3}	$a^2\rho^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\rho^2\sigma_x^2$
ax_{t-4}	$a^2\rho^3\sigma_x^2$	$a^2\rho^2\sigma_x^2$	$a^2\rho\sigma_x^2$	$a^2\sigma_x^2$	$a^2\rho\sigma_x^2$

Table 7: Covariance Terms of X and Λ: $K = 5$

	$a\epsilon_t$	$a\epsilon_{t-1}$	$a\epsilon_{t-2}$	$a\epsilon_{t-3}$	$a\epsilon_{t-4}$
ax_t	$a^2\sigma_{x\epsilon}$	$a^2\rho\sigma_{x\epsilon}$	$a^2\rho^2\sigma_{x\epsilon}$	$a^2\rho^3\sigma_{x\epsilon}$	$a^2\rho^4\sigma_{x\epsilon}$
ax_{t-1}	0	$a^2\sigma_{x\epsilon}$	$a^2\rho\sigma_{x\epsilon}$	$a^2\rho^2\sigma_{x\epsilon}$	$a^2\rho^3\sigma_{x\epsilon}$
ax_{t-2}	0	0	$a^2\sigma_{x\epsilon}$	$a^2\rho\sigma_{x\epsilon}$	$a^2\rho^2\sigma_{x\epsilon}$
ax_{t-3}	0	0	0	$a^2\sigma_{x\epsilon}$	$a^2\rho\sigma_{x\epsilon}$
ax_{t-4}	0	0	0	0	$a^2\sigma_{x\epsilon}$

write the covariance as the sum of all the cells as follows:

$$\begin{aligned}\sigma_{XX_{-1}} &= \frac{\sigma_x^2}{K^2}\left(\rho K + \left(\rho + \frac{1}{\rho}\right)\sum_{i=1}^{K-1}\rho^i (K-i)\right) \\ &= \frac{\sigma_x^2}{K^2}\left(\rho K + \left(\rho + \frac{1}{\rho}\right) M(K,\rho)\right) \qquad (32)\end{aligned}$$

where $M(K,\rho) \equiv \sum_{i=1}^{K-1}\rho^i (K-i)$ and is given in the text in closed form in (16).

The ratio of (32) to (30) is the central function $C(K,\rho)$ given by Equation (15) in the text.

From the table, we see that if $\rho = 0$, then $\sigma_{XX_{-1}} = \frac{\sigma_x^2}{K^2}(K-1)$. If we take the ratio of this expression to the corresponding expression $\sigma_X^2 = \frac{\sigma_x^2}{K}$ found above for $\rho = 0$, we get $C(K,0) = \frac{K-1}{K} = 1 - \frac{1}{K}$.

A.3 The Covariance of X and Λ

Now consider $\sigma_{X\Lambda}$, the covariance of the averaged terms X and Λ . Again, it is useful to appeal to a helper matrix like that in Table 7. The covariance we are looking for is the sum of all of the terms in the matrix. Each entry is given by the following:

1. $Cov(x_{t-i}, \epsilon_{t-i}) = \sigma_{x\epsilon}$ for $i \geq 0$

2. $Cov\left(x_{t-i}, \epsilon_t\right) = 0$ for $i > 0$

3. $Cov\left(x_{t+i}, \epsilon_t\right) = \rho^i \sigma_{x\epsilon}$ for $i > 0$

We know, for example, that $Cov\left(x_{t-i}, \epsilon_t\right) = 0$ because ϵ_t only affects x_t and *later* realizations of x. On the other hand, $Cov\left(x_{t+1}, \epsilon_t\right) = \rho\sigma_{x\epsilon} < 0$ because ϵ_t affects x_t, which then affects x_{t+1} with magnitude ρ. For terms two years apart, we have $Cov\left(x_{t+2}, \epsilon_t\right) = \rho^2\sigma_{x\epsilon} < 0$.

Summing the terms yields:

$$\sigma_{X\Lambda} = \frac{\sigma_{x\epsilon}}{K^2}\left(K + \sum_{i=1}^{K-1} \rho^i \left(K - i\right)\right) = \frac{\sigma_{x\epsilon}}{K^2}\left(K + M\left(K, \rho\right)\right) < 0$$

where $M\left(K, \rho\right)$ is given in closed form in (16).

From the table, we see that if $\rho = 0$, then $\sigma_{X\Lambda} = \frac{\sigma_{x\epsilon}}{K^2}K = \frac{\sigma_{x\epsilon}}{K}$. So taking the ratio with $\sigma_X^2 = \frac{\sigma_x^2}{K}$ in the case of $\rho = 0$, yields $\frac{\sigma_{x\epsilon}}{\sigma_x^2}$.